AP® Calculus AB
AP® Calculus BC

Free-Response Questions and Solutions 1969 – 1978

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Notes About AP Calculus Free-Response Questions

- The solution to each free-response question is based on the scoring guidelines from the AP Reading. Where appropriate, modifications have been made by the editor to clarify the solution. Other mathematically correct solutions are possible.

- Scientific calculators were permitted, but not required, on the AP Calculus Exams in 1983 and 1984.

- Scientific (nongraphing) calculators were required on the AP Calculus Exams in 1993 and 1994.

- Graphing calculators have been required on the AP Calculus Exams since 1995. From 1995 to 1999, the calculator could be used on all six free-response questions. Since the 2000 exams, the free-response section has consisted of two parts -- Part A (questions 1-3) requires a graphing calculator and Part B (questions 4-6) does not allow the use of a calculator.

- Always refer to the most recent edition of the Course Description for AP Calculus AB and BC for the most current topic outline, as earlier exams may not reflect current exam topics.

1969 AB1

Consider the following functions defined for all x:

$f_1(x) = x$
$f_2(x) = x\cos x$
$f_3(x) = 3e^{2x}$
$f_4(x) = x - |x|$

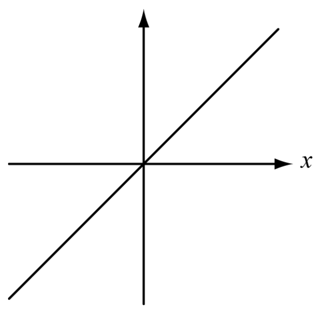
Answer the following questions (a, b, c, and d) about each of these functions. Indicate your answer by writing either yes or no in the appropriate space in the given rectangular grid. No justification is required but each blank space will be scored as an incorrect answer.

	Questions	Functions			
		f_1	f_2	f_3	f_4
(a)	Does $f(-x) = -f(x)$				
(b)	Does the inverse function exist for all x?				
(c)	Is the function periodic?				
(d)	Is the function continuous at $x = 0$?				

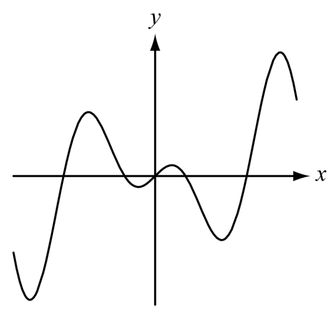
1969 AB1
Solution

	Questions	f_1	f_2	f_3	f_4
(a)	Does $f(-x) = -f(x)$	Yes	Yes	No	No
(b)	Does the inverse function exist for all x?	Yes	No	**	No
(c)	Is the function periodic?	No	No	No	No
(d)	Is the function continuous at $x = 0$?	Yes	Yes	Yes	Yes

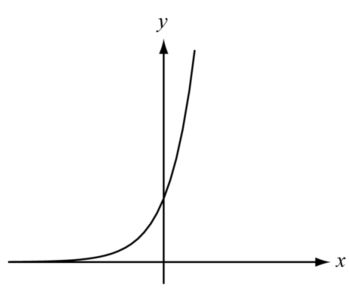
$f_1(x) = x$



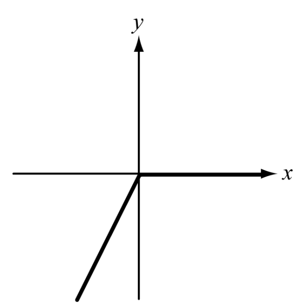
$f_2(x) = x\cos x$



$f_3(x) = 3e^{2x}$



$f_4(x) = x - |x|$



** It is not clear how this question was meant to be interpreted and answered for the function f_3 during the 1969 exam. Reasonable arguments can be made for both an answer of "No," and for an answer of "Yes." No student lost credit for either answer.

If one assumes that the question asks if the inverse function exists and has a domain of all real numbers (i.e., exists for all x), then the answer is "No" since the range of f_3 consists only of the positive numbers.

If one assumes that the question simply asks if an inverse function g exists for which $g(f_3(x)) = x$ for all x, then the answer is "Yes."

1969 AB2/BC2

A particle moves along the x-axis in such a way that its position at time t is given by $x = 3t^4 - 16t^3 + 24t^2$ for $-5 \le t \le 5$.

(a) Determine the velocity and acceleration of the particle at time t.

(b) At what values of t is the particle at rest?

(c) At what values of t does the particle change direction?

(d) What is the velocity when the acceleration is first zero?

1969 AB2/BC2
Solution

(a) $v = \dfrac{dx}{dt} = 12t^3 - 48t^2 + 48t = 12t(t^2 - 4t + 4) = 12t(t-2)^2$

$a = \dfrac{dv}{dt} = 36t^2 - 96t + 48 = 12(3t^2 - 8t + 4) = 12(3t-2)(t-2)$

(b) The particle is at rest when $v = 0$. This occurs when $t = 0$ and $t = 2$.

(c) The particle changes direction at $t = 0$ only.

(d) $a = 0$ when $t = \dfrac{2}{3}$ and $t = 2$. The acceleration is first zero at $t = \dfrac{2}{3}$.

$v\left(\dfrac{2}{3}\right) = 12\left(\dfrac{2}{3}\right)\left(-\dfrac{4}{3}\right)^2 = \dfrac{128}{9}$

1969 AB3/BC3

Given $f(x) = \frac{1}{x} + \ln x$, defined only on the closed interval $\frac{1}{e} \le x \le e$.

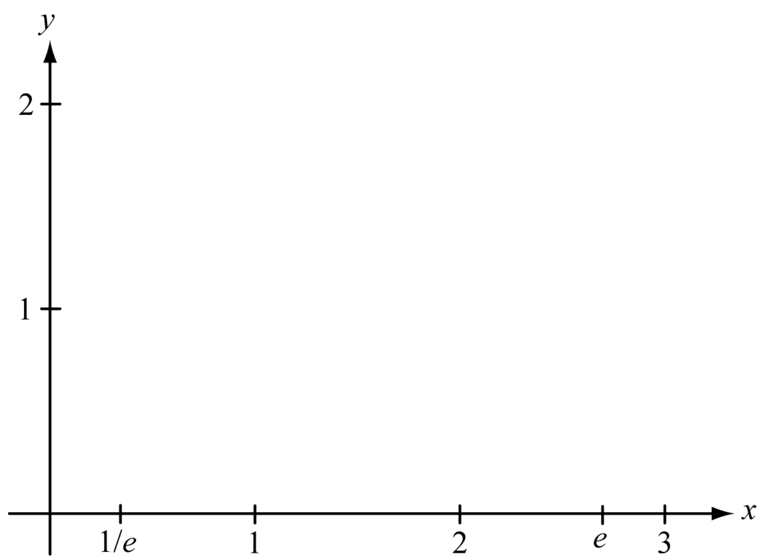
(a) Showing your reasoning, determine the value of x at which f has its

(i) absolute maximum,
(ii) absolute minimum.

(b) For what values of x is the curve concave up?

(c) On the coordinate axes provided, sketch the graph of f over the interval $\frac{1}{e} \le x \le e$.

(d) Given that the mean value (average ordinate) of f over the interval is $\frac{2}{e-1}$, state in words a geometrical interpretation of this number relative to the graph.



1969 AB3/BC3
Solution

(a) $f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}$

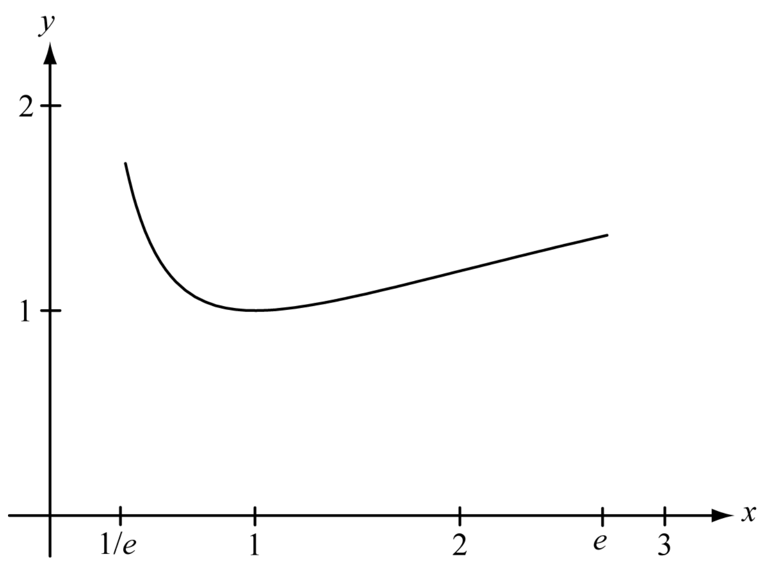
$f'(x) = 0$ at $x = 1$.

The three candidates are
$f(1) = 1$
$f\left(\frac{1}{e}\right) = e - 1$
$f(e) = \frac{1}{e} + 1 = \frac{e+1}{e}$
Therefore the absolute maximum is at $x = \frac{1}{e}$ and the absolute minimum is at $x = 1$.

(b) $f''(x) = \frac{2}{x^3} - \frac{1}{x^2} = \frac{2-x}{x^3} > 0$ for $\frac{1}{e} \le x < 2$.

Therefore the curve is concave up for $\frac{1}{e} \le x < 2$.

(c)



(d) $\frac{2}{e-1}$ is the height of a rectangle of width $\left(e - \frac{1}{e}\right)$ and having the same area as that enclosed by the graph of $y = f(x)$, the vertical lines $x = \frac{1}{e}$ and $x = e$, and the x-axis.

1969 AB4/BC4

The number of bacteria in a culture at time t is given approximately by $y = 1000(25 + te^{-t/20})$ for $0 \le t \le 100$.

(a) Find the largest number and the smallest number of bacteria in the culture during the interval.

(b) At what time during the interval is the rate of change in the number of bacteria a minimum?

1969 AB4/BC4
Solution

(a) $y' = 1000\left(-\frac{t}{20}e^{-t/20} + e^{-t/20}\right) = 1000e^{-t/20}\left(1-\frac{t}{20}\right)$

$y' = 0$ at $t = 20$

At $t = 0$, $y = 1000(25)$
At $t = 20$, $y = 1000(25 + 20e^{-1})$
At $t = 100$, $y = 1000(25 + 100e^{-5})$

The minimum number of bacteria is $1000(25)$.
Since $20e^{-1} > 100e^{-5}$ because $e^4 > 5$, the maximum number is $1000(25 + 20e^{-1})$.

(b) $y'' = 1000e^{-t/20}\left(-\frac{1}{20} + \left(1-\frac{t}{20}\right)\left(-\frac{1}{20}\right)\right) = \frac{1000}{20}e^{-t/20}\left(-1-1+\frac{t}{20}\right)$
$= 50e^{-t/20}\left(\frac{t}{20} - 2\right)$

$y'' = 0$ at $t = 40$

At $t = 0$, $y' = 1000$
At $t = 40$, $y' = 1000(-e^{-2})$
At $t = 100$, $y' = 1000(-4e^{-5})$

The rate of change is a minimum at $t = 40$.

1969 AB5

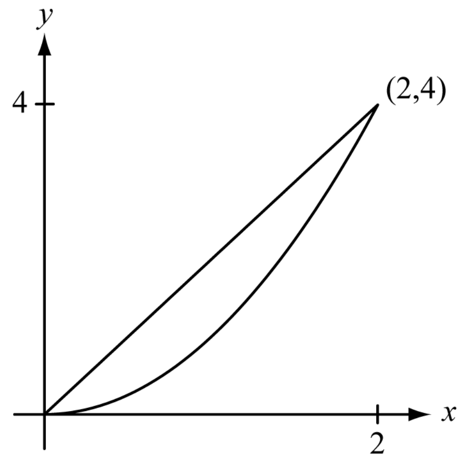
Let R denote the region enclosed between the graph of $y = x^2$ and the graph of $y = 2x$.

(a) Find the area of region R.

(b) Find the volume of the solid obtained by revolving the region R about the y-axis.

1969 AB5
Solution

Intersection: $x^2 = 2x$ when $x = 0$ and $x = 2$



(a) Area $= \int_0^2 (2x - x^2)\,dx$

$$= x^2 - \frac{x^3}{3}\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

or

Area $= \int_0^4 \left(y^{1/2} - \frac{y}{2} \right) dy$

$$= \frac{2}{3} y^{3/2} - \frac{y^2}{4}\Big|_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$$

(b) Shells:

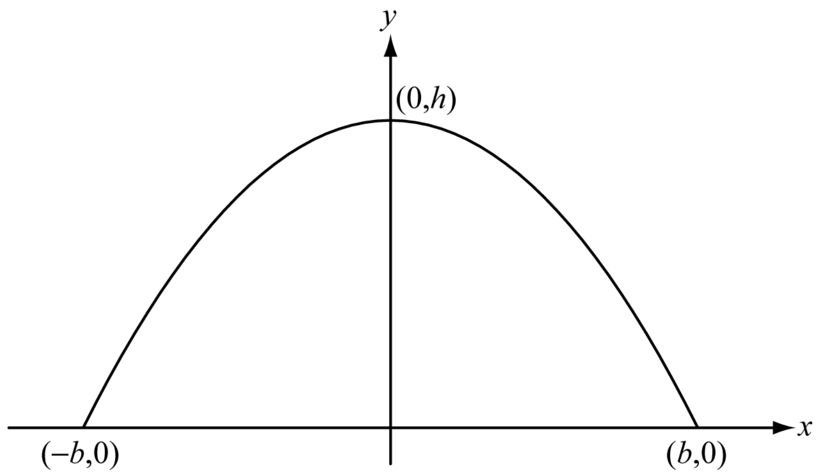
$$\text{Volume} = 2\pi \int_0^2 x(2x - x^2)\,dx = 2\pi \left(\frac{2}{3} x^3 - \frac{x^4}{4} \right)\Bigg|_0^2 = 2\pi \left(\frac{16}{3} - 4 \right) = \frac{8}{3}\pi$$

Disks:

$$\text{Volume} = \pi \int_0^4 \left(\left(y^{1/2} \right)^2 - \left(\frac{y}{2} \right)^2 \right) dy = \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy$$

$$= \pi \left(\frac{y^2}{2} - \frac{y^3}{12} \right)\Bigg|_0^4 = \pi \left(8 - \frac{64}{12} \right) = \frac{8}{3}\pi$$

1969 AB6



An arched window with base width $2b$ and height h is set into a wall. The arch is to be either an arc of a parabola or a half-cycle of a cosine curve.

(a) If the arch is an arc of a parabola, write an equation for the parabola relative to the coordinate system shown in the figure.

(b) If the arch is a half-cycle of a cosine curve, write an equation for the cosine curve relative to the coordinate system shown in the figure.

(c) Of these two window designs, which has the greater area? Justify your answer.

1969 AB6
Solution

(a) $y = Ax^2 + Bx + C$

$$h = A \cdot 0^2 + B \cdot 0 + C = C$$

$$\left.\begin{aligned} 0 &= Ab^2 + Bb + C = Ab^2 + Bb + h \\ 0 &= Ab^2 - Bb + C = Ab^2 - Bb + h \end{aligned}\right\} \Rightarrow B = 0$$

$$Ab^2 + h = 0 \Rightarrow A = -\frac{h}{b^2}$$

Therefore $y = -\frac{h}{b^2}x^2 + h = \frac{h}{b^2}(b^2 - x^2)$.

The parabola could also be written as $x^2 = -\frac{b^2}{h}(y - h)$.

(b) $y = A\cos Bx$

$$h = A\cos 0 = A$$

$$\left.\begin{aligned} 0 &= A\cos Bb \\ 0 &= A\cos B(-b) = A\cos Bb \end{aligned}\right\} Bb = \frac{\pi}{2} \Rightarrow B = \frac{\pi}{2b}$$

Therefore $y = h\cos\left(\frac{\pi x}{2b}\right)$

(c) For parabola, area $= 2\int_0^b \frac{h}{b^2}(b^2 - x^2)\,dx = 2\frac{h}{b^2}\left(b^2 x - \frac{x^3}{3}\right)\Bigg|_0^b = \frac{2h}{b^2}\left(\frac{2}{3}b^3\right) = \frac{4hb}{3}$

For cosine design, area $= 2\int_0^b h\cos\left(\frac{\pi x}{2b}\right)dx = 2h\left(\frac{2b}{\pi}\right)\sin\left(\frac{\pi x}{2b}\right)\Bigg|_0^b = \frac{4bh}{\pi}$

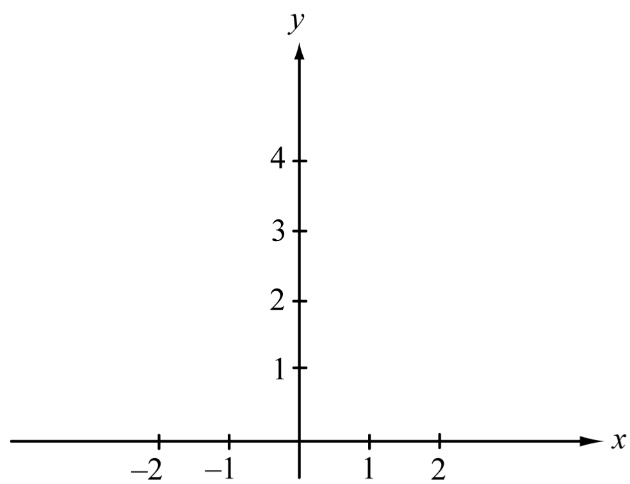
The parabola design has the greater area since $3 < \pi$.

1969 AB7

(a) On the coordinate axes provided, sketch the graph of $y = \dfrac{e^x + e^{-x}}{2}$.

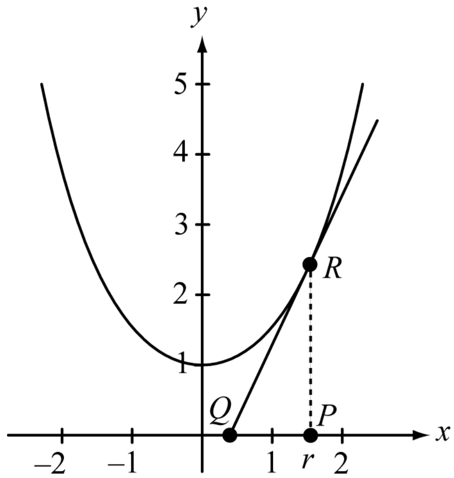
(b) Let R be a point on the curve and let the x-coordinate of R be r $(r \neq 0)$. The tangent line to the curve at R crosses the x-axis at a point Q. Find the coordinates of Q.

(c) If P is the point $(r, 0)$, find the length of PQ as a function of r and the limiting value of this length as r increases without bound.



1969 AB7
Solution

(a)



(b) Method 1

$y' = \dfrac{e^x - e^{-x}}{2}$

The equation of the tangent line at the point R is

$$y - \frac{e^r + e^{-r}}{2} = \frac{e^r - e^{-r}}{2}(x - r) \text{ or}$$

$$y = \frac{e^r - e^{-r}}{2}x + \frac{e^r + e^{-r}}{2} - \frac{e^r - e^{-r}}{2}r.$$

The line crosses the x-axis when $y = 0$.

This happens at $x = r - \dfrac{e^r + e^{-r}}{e^r - e^{-r}}$.

The coordinates of Q are $\left(r - \dfrac{e^r + e^{-r}}{e^r - e^{-r}}, 0\right)$.

Method 2 (hyperbolic functions)

$y = \cosh(x)$
$y' = \sinh(x)$

The equation of the tangent line at the point R is

$y - \cosh(r) = \sinh(r) \cdot (x - r)$ or
$y = \sinh(r) \cdot x + \cosh(x) - r\sinh(r)$

The line crosses the x-axis when $y = 0$. This happens at $x = r - \coth(r)$.

The coordinates of Q are $(r - \coth(r), 0)$.

(c) Method 1

$$\overline{PQ} = \frac{e^r + e^{-r}}{e^r - e^{-r}} = \frac{1 + e^{-2r}}{1 - e^{-2r}}$$

$$\lim_{r \to \infty} \overline{PQ} = \lim_{r \to \infty} \frac{1 + e^{-2r}}{1 - e^{-2r}} = 1$$

Method 2

$$\overline{PQ} = \coth(r)$$

$$\lim_{r \to \infty} \overline{PQ} = \lim_{r \to \infty} \coth(r) = 1$$

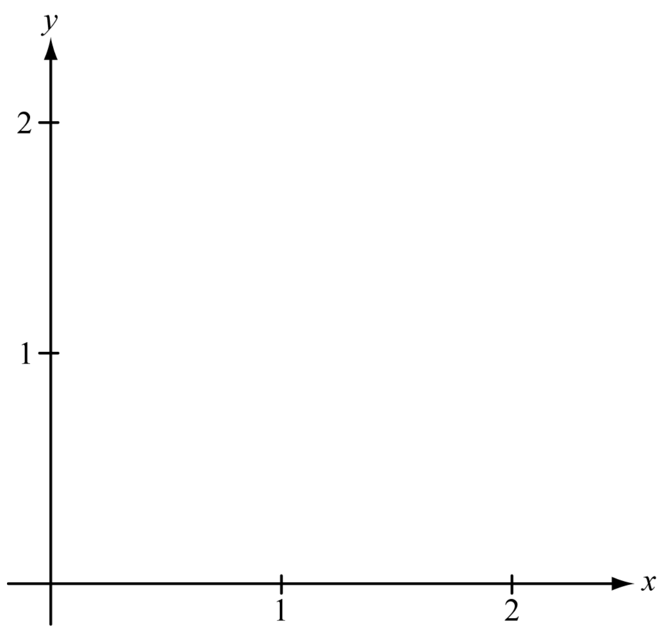
1969 BC1

A closed region R of the plane has $y = 1 + \sin\left(\frac{\pi x}{2}\right)$ as its upper boundary, $y = \frac{x}{2}$ as its lower boundary, and the y-axis as its left-hand boundary.

(a) Sketch the region R on the axes provided.

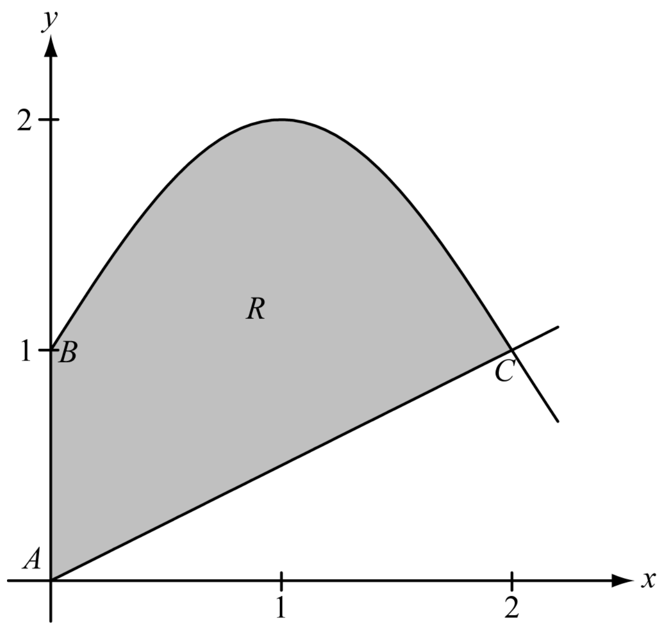
(b) Set up but do not evaluate an integral expression in terms of the single variable x for each of the following:

(i) the area A of R,
(ii) the volume V of the solid obtained by revolving R about the x-axis,
(iii) the total perimeter P of R.



1969 BC1
Solution

(a)



(b) (i) Area $= \int_0^2 \left(1+\sin\left(\frac{\pi x}{2}\right)-\frac{x}{2}\right)dx$

(ii) Volume $= \pi\int_0^2 \left(\left(1+\sin\left(\frac{\pi x}{2}\right)\right)^2-\left(\frac{x}{2}\right)^2\right)dx$

(iii) Let $y_1 = 1+\sin\left(\frac{\pi x}{2}\right)$ be the curve for the upper boundary. Then

$y_1{}' = \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$. The total perimeter is

$$P = \overline{AC}+\overline{AB}+\widehat{BC} = \sqrt{5}+1+\int_0^2 \sqrt{1+\left(\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\right)^2}\,dx$$

1969 BC5

(a) Prove that the area of the region in the first quadrant between the curve $y = e^{-x}$ and the x-axis is divided into two equal parts by the line $x = \ln 2$.

(b) If the two regions of equal area described in part (a) are rotated about the x-axis, are the resulting volumes equal? If so, prove it. If not, determine which is larger and by how much.

1969 BC5
Solution

(a) Method 1

$$A_{\text{Left}} = \int_0^{\ln 2} e^{-x}\,dx = -e^{-x}\Big|_0^{\ln 2} = 1 - e^{-\ln 2} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$A_{\text{Right}} = \int_{\ln 2}^{\infty} e^{-x}\,dx = \lim_{R\to\infty}\int_{\ln 2}^{R} e^{-x}\,dx = \lim_{R\to\infty}(-e^{-x})\Big|_{\ln 2}^{R} = \lim_{R\to\infty}(-e^{-R} + e^{-\ln 2}) = \frac{1}{2}$$

Hence the line $x = \ln 2$ divides the region into two parts of area both equal to $\frac{1}{2}$.

Method 2

$$A_{\text{Left}} = \int_0^{k} e^{-x}\,dx = -e^{-x}\Big|_0^{k} = 1 - e^{-k}$$

$$A_{\text{Right}} = \int_{k}^{\infty} e^{-x}\,dx = \lim_{R\to\infty}\int_{k}^{R} e^{-x}\,dx = \lim_{R\to\infty}(-e^{-x})\Big|_{k}^{R} = \lim_{R\to\infty}(-e^{-R} + e^{-k}) = e^{-k}$$

The two areas will be equal if $1 - e^{-k} = e^{-k}$.

$2e^{-k} = 1 \Rightarrow e^{k} = 2 \Rightarrow k = \ln 2$

Hence the line $x = \ln 2$ divides the region into two parts of equal area.

(b) $$V_{\text{Left}} = \pi\int_0^{\ln 2}(e^{-x})^2\,dx = \pi\left(-\frac{1}{2}e^{-2x}\right)\Big|_0^{\ln 2} = \frac{\pi}{2}(1 - e^{-2\ln 2}) = \frac{\pi}{2}\left(1 - \frac{1}{4}\right) = \frac{3\pi}{8}$$

$$V_{\text{Right}} = \pi\int_{\ln 2}^{\infty}(e^{-x})^2\,dx = \pi\lim_{R\to\infty}\int_{\ln 2}^{R} e^{-2x}dx$$

$$= \pi\lim_{R\to\infty}\left(-\frac{e^{-2x}}{2}\right)\Big|_{\ln 2}^{R} = \frac{\pi}{2}\lim_{R\to\infty}\left(-e^{-2R} + e^{-2\ln 2}\right) = \frac{\pi}{8}$$

The volumes are not equal. The volume of the left region is larger by the amount $V_{\text{Left}} - V_{\text{Right}} = \frac{\pi}{4}$.

1969 BC6

Assume that the graph of a function f passes through the origin and has slope 2 at that point.

(a) Determine $f(x)$ if $f''(x) + 2f'(x) = 0$ for all x.

(b) Determine $f(x)$ if $f''(x) + 2f'(x) = 2$ for all x.

1969 BC6
Solution

(a) $m^2 + 2m = 0 \Rightarrow m = 0, -2$

The general solution is $f(x) = C_1 + C_2 e^{-2x}$. Then $f'(x) = -2C_2 e^{-2x}$. Since the function passes through the origin with slope 2,

$$0 = f(0) = C_1 + C_2$$
$$2 = f'(0) = -2C_2$$

Therefore $C_2 = -1$ and $C_1 = 1$.

$f(x) = 1 - e^{-2x}$

(b) We need to find a particular solution. Since a constant function is already a solution to the homogeneous equation given in part (a), we try a function of the form $y_p = Ax$. Substituting into the differential equation gives

$$0 + 2A = 2$$

and so $A = 1$. Hence $y_p = x$ and the general solution to the differential equation is $f\left(x\right) = C_1 + C_2 e^{-2x} + x$.

Using the initial conditions,

$$\left.\begin{array}{l} 0 = f(0) = C_1 + C_2 \\ 2 = f'(0) = -2C_2 + 1 \end{array}\right\} \Rightarrow C_2 = -\frac{1}{2},\ C_1 = \frac{1}{2}$$

Hence $f(x) = \frac{1}{2}(1 - e^{-2x}) + x$

(Note: there are other methods that can be used to find a particular solution or the general solution, such as variation of parameters or reduction of order.)

1969 BC7

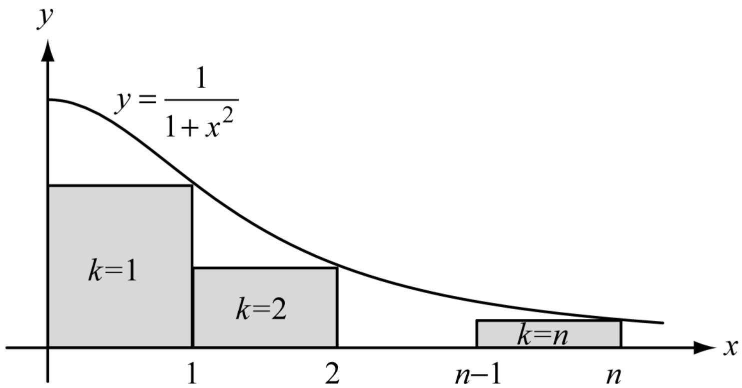
Given the finite sum $S_n = \sum_{k=1}^{n} \frac{1}{1+k^2}$.

(a) By comparing S_n with an appropriate integral, prove that $S_n \le \arctan n$ for $n \ge 1$.

(b) Use part (a) to deduce that $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$ exists. Show your reasoning.

(c) Prove that $\frac{\pi}{4} \le \sum_{k=1}^{\infty} \frac{1}{1+k^2} \le \frac{\pi}{2}$.

1969 BC7
Solution



(a) S_n is a lower (right) Riemann sum for $\int_0^n \frac{dx}{1+x^2}dx$. Therefore

$$S_n \le \int_0^n \frac{dx}{1+x^2} = \arctan n.$$

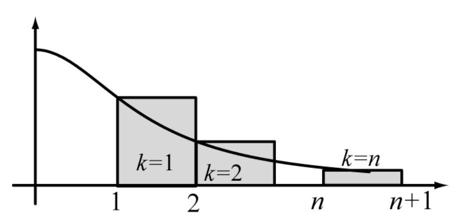
(b) S_n is monotone increasing (by inspection).

$$\int_0^\infty \frac{dx}{1+x^2}dx = \lim_{n\to\infty}\int_0^n \frac{dx}{1+x^2}dx = \lim_{n\to\infty} \arctan n = \frac{\pi}{2}$$

So by part (a), the sequence (S_n) is bounded above. Since the sequence is increasing and bounded, $\sum_{k=1}^{\infty}\frac{1}{1+k^2} = \lim_{n\to\infty} S_n$ exists.

(c) By considering S_n as an upper (left) Riemann sum and also using part (a),

$$\int_1^{n+1} \frac{dx}{1+x^2}dx \le S_n \le \int_0^n \frac{dx}{1+x^2}dx.$$



Therefore $\arctan(n+1) - \frac{\pi}{4} \le S_n \le \arctan n$.

$$\lim_{n\to\infty}\left(\arctan(n+1) - \frac{\pi}{4}\right) \le \lim_{n\to\infty} S_n \le \lim_{n\to\infty} \arctan n.$$

Hence $\frac{\pi}{2} - \frac{\pi}{4} \le \sum_{k=1}^{\infty}\frac{1}{1+k^2} \le \frac{\pi}{2}$ which gives $\frac{\pi}{4} \le \sum_{k=1}^{\infty}\frac{1}{1+k^2} \le \frac{\pi}{2}$.

1970 AB1/BC1

Given the parabola $y = x^2 - 2x + 3$:

(a) Find an equation for the line L, which contains the point $(2,3)$ and is perpendicular to the line tangent to the parabola at $(2,3)$.

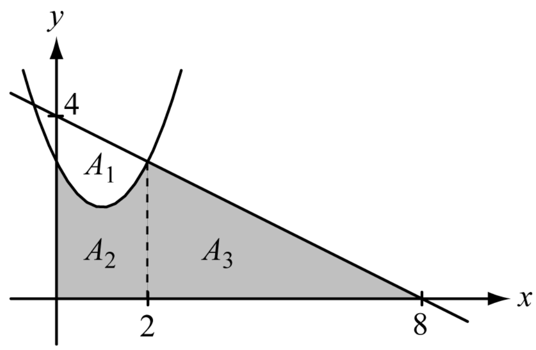
(b) Find the area of that part of the first quadrant which lies below both the line L and the parabola.

1970 AB1/BC1
Solution

(a) $y' = 2x - 2$

The slope of the line tangent to the parabola is $m = 2$. Therefore the slope of the line L that is perpendicular to the tangent line is $-\frac{1}{2}$. The equation of the line L is $y - 3 = -\frac{1}{2}(x - 2)$, or $y = -\frac{1}{2}x + 4$, or $x + 2y = 8$.

(b)



$$A_1 = \int_0^2 \left(\left(-\frac{1}{2}x + 4 \right) - (x^2 - 2x + 3) \right) dx = \int_0^2 \left(1 + \frac{3}{2}x - x^2 \right) dx$$

$$= \left(x + \frac{3}{4}x^2 - \frac{1}{3}x^3 \right) \Bigg|_0^2 = \frac{7}{3}$$

$$A_2 = \int_0^2 (x^2 - 2x + 3)\, dx = \left(\frac{1}{3}x^3 - x^2 + 3x \right) \Bigg|_0^2 = \frac{14}{3}$$

$$A_3 = \int_2^8 \left(-\frac{1}{2}x + 4 \right) dx = 9 \quad \text{(or } A_3 = \frac{1}{2} \cdot 6 \cdot 3 = 9 \text{ using area of the triangle)}$$

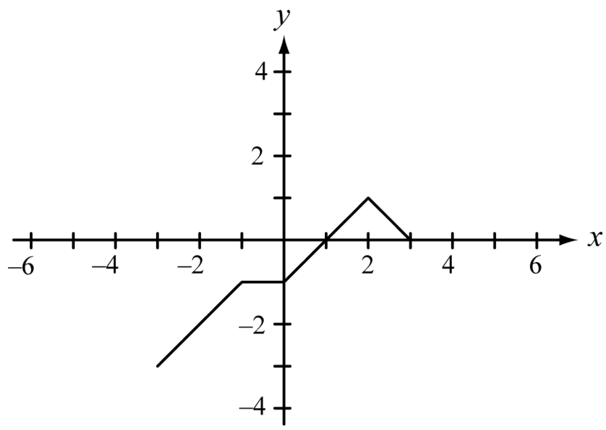
Area of triangle $(A_1 + A_2 + A_3)$ is $\frac{1}{2} \cdot 8 \cdot 4 = 16$

Method 1: area $= A_2 + A_3 = \frac{41}{3}$

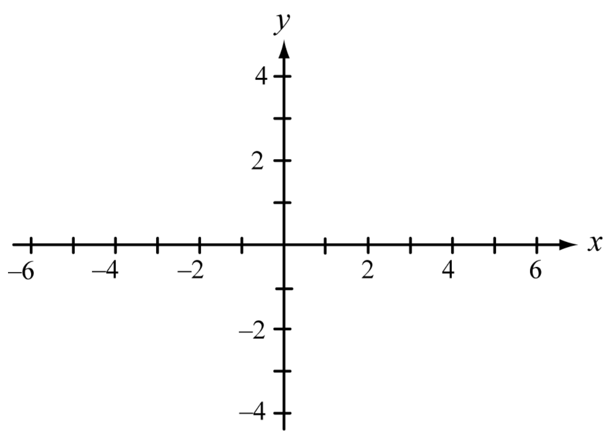
Method 2: area $= 16 - A_1 = 16 - \frac{7}{3} = \frac{41}{3}$

1970 AB2

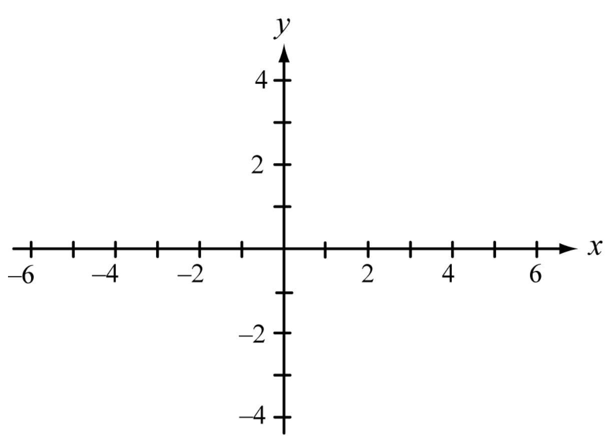
A function f is defined on the closed interval from -3 to 3 and has the graph shown below.



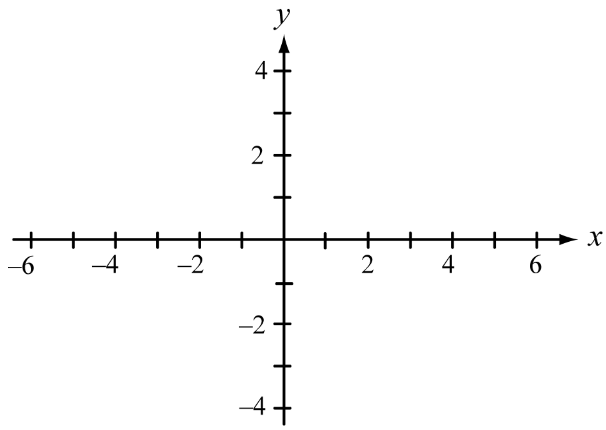
(a) On the axes provided sketch the entire graph of $y = |f(x)|$.



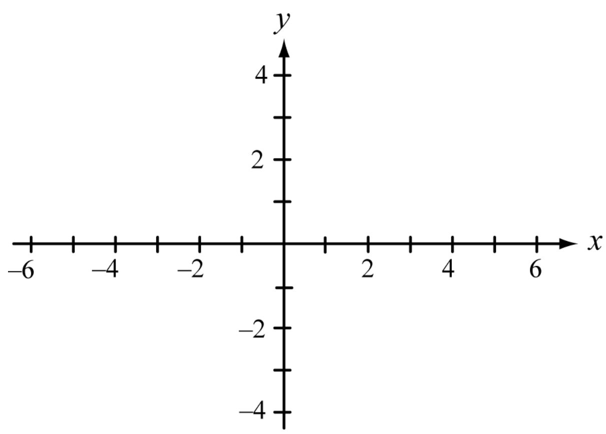
(b) On the axes provided sketch the entire graph of $y = f(|x|)$.



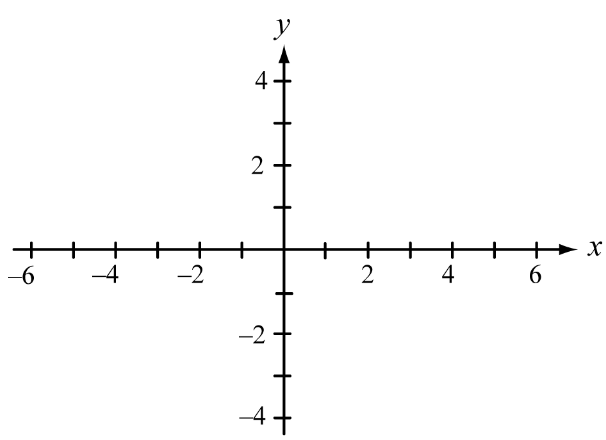
(c) On the axes provided sketch the entire graph of $y = f(-x)$.



(d) On the axes provided sketch the entire graph of $y = f\left(\frac{1}{2}x\right)$.

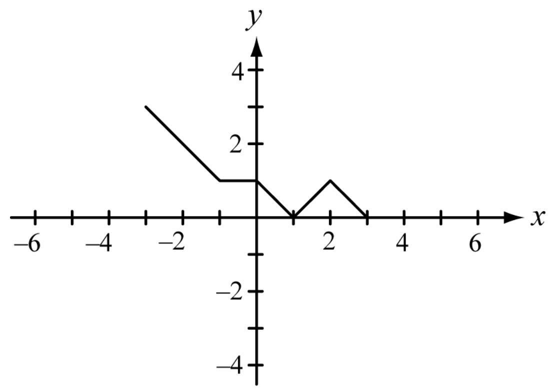


(e) On the axes provided sketch the entire graph of $y = f(x-1)$.

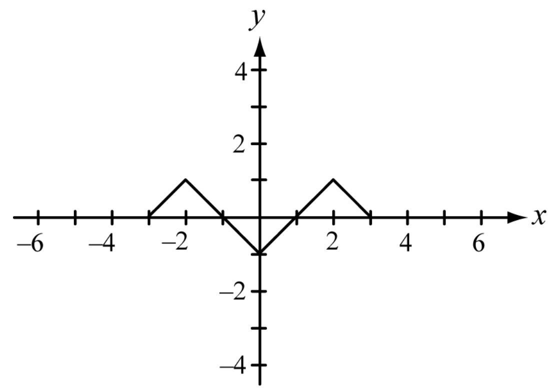


1970 AB2
Solution

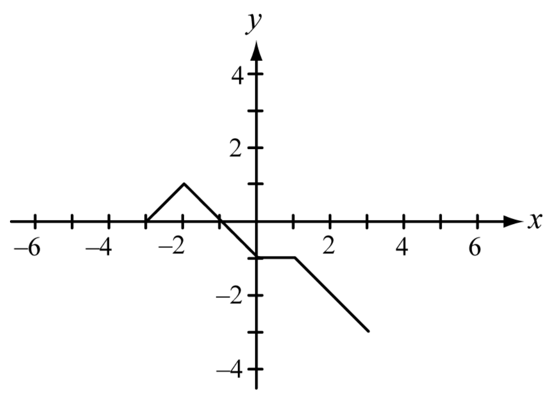
(a) $y = |f(x)|$



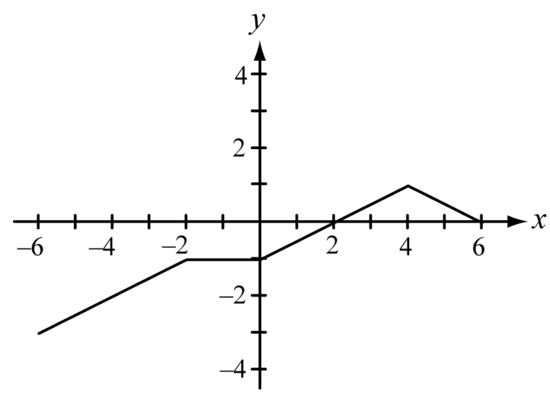
(b) $y = f(|x|)$



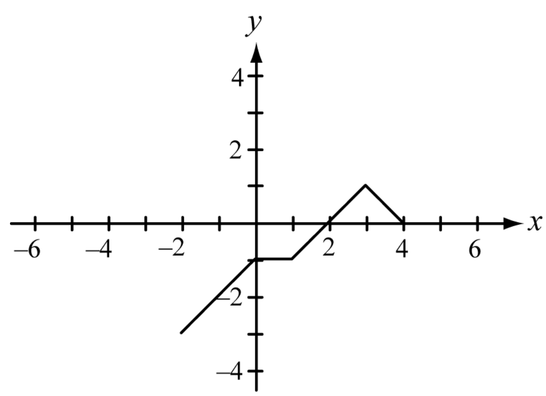
(c) $y = f(-x)$



(d) $y = f\left(\frac{1}{2}x\right)$



(e) $y = f(x-1)$



1970 AB3/BC2

Consider the function f given by $f(x) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}}$ on the interval $-8 \le x \le 8$.

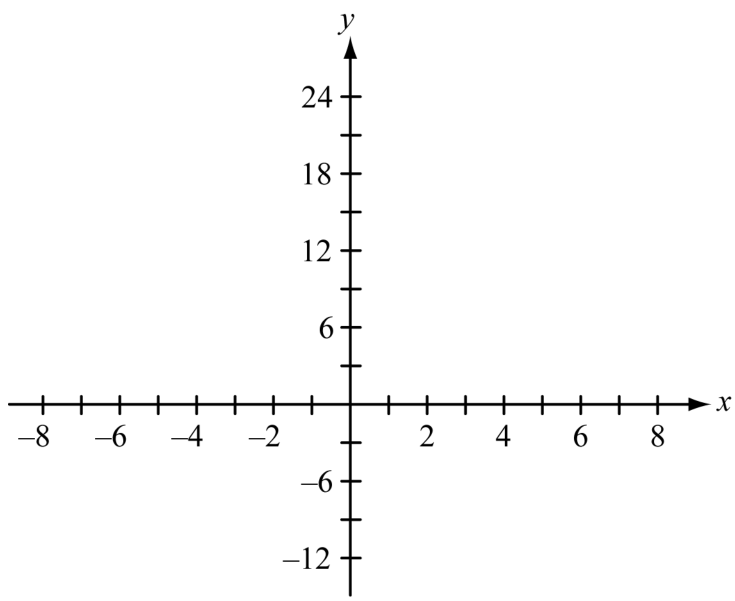
(a) Find the coordinates of all points at which the tangent to the curve is a horizontal line.

(b) Find the coordinates of all points at which the tangent to the curve is a vertical line.

(c) Find the coordinates of all points at which the absolute maximum and absolute minimum occur.

(d) For what values of x is this function concave down?

(e) On the axes provided, sketch the graph of the function on this interval.



1970 AB3/BC2
Solution

(a) $f(x) = x^{4/3} + 4x^{1/3} = x^{1/3}(x+4)$

$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}\left(\frac{x+1}{x^{2/3}}\right)$

$f'(x) = 0$ at $x = -1$. There is a horizontal tangent at $(-1,-3)$.

(b) There is a vertical tangent at $(0,0)$.

(c) The absolute maximum and absolute minimum must occur at a critical point or an endpoint. The candidates are

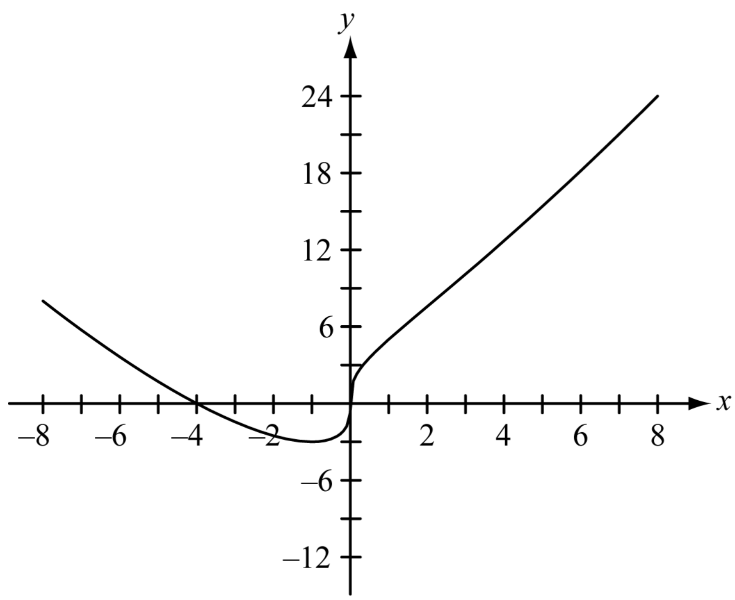
$(-8,8)$, $(-1,-3)$, $(0,0)$, and $(8,24)$

So the absolute maximum is at $(8,24)$ and the absolute minimum is at $(-1,-3)$.

(d) $f''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}\left(\frac{x-2}{x^{5/3}}\right)$

The graph is concave down for $0 < x < 2$.

(e)



1970 AB4

A right circular cone and a hemisphere have the same base, and the cone is inscribed in the hemisphere. The figure is expanding in such a way that the combined surface area of the hemisphere and its base is increasing at a constant rate of 18 square inches per second. At what rate is the volume of the cone changing at the instant when the radius of the common base is 4 inches? Show your work.

<u>Note:</u> The surface area of a sphere of radius r is $S = 4\pi r^2$ and the volume of a right circular cone of height h and base radius r is $V = \frac{1}{3}\pi r^2 h$.

1970 AB4
Solution

Method 1:

The combined surface area of the hemisphere and its base is
$S = \frac{1}{2}(4\pi r^2) + \pi r^2 = 3\pi r^2$

$18 = \frac{dS}{dt} = 6\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{3}{\pi r}$

Since the height of the cone is $h = r$, the volume of the cone is $V = \frac{1}{3}\pi r^3$

$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt} = \pi r^2 \left(\frac{3}{\pi r}\right) = 3r$

At $r = 4$, $\frac{dV}{dt} = 12$

Method 2:

As in method 1, $S = 3\pi r^2$ and so $V = \frac{1}{3}\pi\left(\frac{S}{3\pi}\right)^{\frac{3}{2}}$.

$\frac{dV}{dt} = \frac{1}{3}\pi \cdot \frac{3}{2}\left(\frac{S}{3\pi}\right)^{\frac{1}{2}} \cdot \frac{1}{3\pi} \cdot \frac{dS}{dt} = \frac{1}{6}\left(\frac{S}{3\pi}\right)^{\frac{1}{2}} \frac{dS}{dt}$

When $r = 4$, $S = 48\pi$ and so $\frac{dV}{dt} = \frac{1}{6} \cdot 4 \cdot 18 = 12$

1970 AB5

A particle moves along the x-axis in such a way that at time $t > 0$ its position coordinate is $x = \sin(e^t)$.

(a) Find the velocity and acceleration of the particle at time t.

(b) At what time does the particle first have zero velocity?

(c) What is the acceleration of the particle at the time determined in part (b)?

1970 AB5
Solution

(a) $x = \sin(e^t)$

$$v = \frac{dx}{dt} = e^t \cos(e^t)$$

$$a = \frac{dv}{dt} = e^t(\cos(e^t) - e^t \sin(e^t))$$

(b) $v(t) = 0$ when $\cos(e^t) = 0$. Hence $e^t = \frac{\pi}{2}$ gives the first time when the velocity is zero, and so $t = \ln\frac{\pi}{2}$.

(c) $a\left(\ln\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}\right) = -\frac{\pi^2}{4}$

1970 AB6/BC5

A parabola P is symmetric to the y-axis and passes through $(0,0)$ and $(b,\, e^{-b^2})$ where $b > 0$.

(a) Write an equation for P.

(b) The closed region bounded by P and the line $y = e^{-b^2}$ is revolved about the y-axis to form a solid figure F. Compute the volume of F.

(c) For what value of b is the volume of F a maximum? Justify your answer.

1970 AB6/BC5
Solution

(a) $y = Ax^2 + Bx + C$.

Since $y = 0$ when $x = 0$, must have $C = 0$.
Since the graph is symmetric to the y-axis, must have $B = 0$.

Since $e^{-b^2} = Ab^2$, must have $A = \frac{e^{-b^2}}{b^2}$.

Therefore $y = \frac{x^2}{b^2}e^{-b^2}$ or $x^2 = yb^2e^{b^2}$

(b) <u>Disks</u>:

$$\text{Volume} = \pi\int_0^{e^{-b^2}} x^2\,dy = \pi\int_0^{e^{-b^2}} yb^2e^{b^2}\,dy = \pi b^2e^{b^2}\left(\frac{y^2}{2}\right)\Bigg|_0^{e^{-b^2}} = \frac{1}{2}\pi b^2e^{-b^2}$$

<u>Shells</u>:

$$\text{Volume} = 2\pi\int_0^b x\left(e^{-b^2} - \frac{x^2}{b^2}e^{-b^2}\right)dx = 2\pi e^{-b^2}\left(\frac{x^2}{2} - \frac{x^4}{4b^2}\right)\Bigg|_0^b = \frac{1}{2}\pi b^2e^{-b^2}$$

(c) $V(b) = \frac{1}{2}\pi b^2e^{-b^2}$

$V'(b) = \pi b(1-b^2)e^{-b^2}$

$V'(b) = 0$ when $b = -1, 0,$ or 1. Since $b > 0$, the only viable solution is $b = 1$.

$V''(b) = \pi e^{-b^2}(2b^4 - 5b^2 + 1) < 0$ when $b = 1$. Therefore there is a local maximum at $b = 1$. Since this is the only critical point for $b > 0$, it is also the absolute maximum for $b > 0$. Another way to justify the absolute maximum at $b = 1$ is to observe that $V'(b) > 0$ for $0 < b < 1$, so that $V(b)$ is increasing on this interval, and $V'(b) < 0$ for $b > 1$, so that $V(b)$ is decreasing for $b > 1$.

1970 AB7

From the fact that $\sin t \le t$ for all $t \ge 0$, use integration repeatedly to prove the following inequalities. Show your work.

$$1-\frac{1}{2!}x^2 \le \cos x \le 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4 \quad \text{for all } x \ge 0$$

1970 AB7
Solution

Given $\sin t \le t$ for all $t \ge 0$, integrate both sides over the interval $0 \le t \le x$ for $x \ge 0$.

$$\int_0^x \sin t\, dt \le \int_o^x t\, dt$$

$$-\cos t\Big|_0^x \le \frac{t^2}{2}\Bigg|_0^x$$

$$-\cos x + 1 \le \frac{x^2}{2}$$

$$\cos x \ge 1 - \frac{x^2}{2} \text{ for } x \ge 0$$

Now integrate both sides with respect to *t* over the interval $0 \le t \le x$ for $x \ge 0$.

$$\int_0^x \cos t\, dt \ge \int_0^x \left(1 - \frac{t^2}{2}\right) dt$$

$$\sin t\,\Big|_0^x \ge \left(t - \frac{t^3}{6}\right)\Bigg|_0^x$$

$$\sin x \ge x - \frac{x^3}{6} \text{ for } x \ge 0$$

Now integrate both sides with respect to *t* over the interval $0 \le t \le x$ for $x \ge 0$.

$$\int_0^x \sin t\, dx \ge \int_0^x \left(t - \frac{t^3}{6}\right) dt$$

$$-\cos t\,\Big|_0^x \ge \left(\frac{t^2}{2} - \frac{t^4}{24}\right)\Bigg|_0^x$$

$$-\cos x + 1 \ge \frac{x^2}{2} - \frac{x^4}{24}$$

$$\cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{24} \text{ for } x \ge 0$$

1970 BC3

A particle moves along the x-axis in such a way that at time $t > 0$, its position coordinate is $x = \sin(e^t)$.

(a) Find the velocity of the particle at time t.

(b) At what time does the particle first have zero velocity?

(c) Show that the length of time between successive instants at which the particle has zero velocity decreases as t increases.

1970 BC3
Solution

(a) $x = \sin(e^t)$

$v = e^t \cos(e^t)$

(b) $v(t) = 0$ when $\cos(e^t) = 0$. Hence $e^t = \frac{\pi}{2}$ gives the first time when the velocity is zero, and so $t = \ln\frac{\pi}{2}$.

(c) $v(t) = 0$ when $e^t = (2n-1)\frac{\pi}{2}$, that is, at $t_n = \ln\left((2n-1)\frac{\pi}{2}\right)$ for each integer *n*. Let $T(n)$ be the time between instants *n* and $n+1$. Then $T(n) = t_{n+1} - t_n = \ln\frac{2n+1}{2n-1}$.

Method 1:

Let $T(x) = \ln\frac{2x+1}{2x-1}$. Then $T'(x) = \frac{-4}{(2x+1)(2x-1)} < 0$ for $x > 1$ and so $T(x)$ is a decreasing function of *x* for $x > 1$. Therefore $T(n)$ decreases as *n* increases.

Method 2:

Since $\frac{2n+1}{2n-1} = 1 - \frac{2}{2n-1}$ is a decreasing function of *n* and $\ln x$ is an increasing function, then the composition $T(n)$ is a decreasing function.

1970 BC4

(a) Find the general solution of the differential equation $\dfrac{d^2y}{dx^2} + 25y = 0$.

(b) Find the general solution of the differential equation $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0$.

(c) For the differential equation $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0$, find the solution that has a graph passing through the point $(0, 2)$ with slope 6.

1970 BC4
Solution

(a) $y'' + 25y = 0$

The auxiliary equation $m^2 + 25 = 0$ has roots $m = \pm 5i$.
The general solution is $y = C_1 \sin 5x + C_2 \cos 5x$

This can also be written as $y = C_1 \sin 5x + C_2 \cos 5x$ or $y = C_5 \cos(5x + C_6)$.

(b) The auxiliary equation $m^2 - 6m + 25 = 0$ has roots $m = 3 \pm 4i$.
The general solution is $y = e^{3x}(C_1 \cos 4x + C_2 \sin 4x)$.

This can also be written as $y = C_3 e^{3x} \sin(4x + C_4)$ or $y = C_5 e^{3x} \cos(4x + C_6)$.

(c) $y = e^{3x}(C_1 \cos 4x + C_2 \sin 4x)$
$y' = e^{3x}((4C_2 + 3C_1)\cos 4x + (-4C_1 + 3C_2)\sin 4x)$

From the given conditions,
$$\begin{cases} 2 = C_1 \\ 6 = 4C_2 + 3C_1 \end{cases}$$
This has the solution $C_1 = 2$ and $C_2 = 0$. Therefore $y = 2e^{3x} \cos 4x$.

1970 BC6

Let $a_k = (-1)^{k+1} \int_0^{\pi/k} \sin kx \, dx$.

(a) Evaluate a_k.

(b) Show that the infinite series $\sum_{k=1}^{\infty} a_k$ converges.

(c) Show that $1 \le \sum_{k=1}^{\infty} a_k \le \frac{3}{2}$.

1970 BC6
Solution

(a) $\int_0^{\pi/k} \sin kx\,dx = -\frac{1}{k}\cos kx\Big|_0^{\pi/k} = \frac{1}{k}(1-\cos\pi) = \frac{2}{k}$, so $a_k = (-1)^{k+1}\frac{2}{k}$

(b) The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{2}{k}\right)$ converges by the Alternating Series Test since

(i) the terms alternate

(ii) the terms are decreasing in absolute value, i.e. $|a_{k+1}| < |a_k|$ for all $k \ge 1$

(iii) $\lim_{k\to\infty} a_k = 0$

Alternatively, the series converges because

(i) $\lim_{k\to\infty} a_k = 0$

(ii) The partial sums S_{2n} are bounded above

(iii) The partial sums S_{2n+1} are bounded below

(c) Method 1

$$\sum_{k=1}^{\infty} a_k = 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{k+1}\left(\frac{1}{k}\right)+\cdots\right)$$

$$= 2\left(\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\right) = 2\left(\frac{1}{2}+\frac{1}{12}+\cdots\right) \ge 2\left(\frac{1}{2}\right) = 1$$

$$\sum_{k=1}^{\infty} a_k = 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{k+1}\left(\frac{1}{k}\right)+\cdots\right)$$

$$= 2\left(1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{6}-\frac{1}{7}\right)-\left(\frac{1}{8}-\frac{1}{9}\right)-\cdots\right)$$

$$= 2\left(1-\frac{1}{6}-\frac{1}{20}-\frac{1}{42}-\frac{1}{72}-\cdots\right) < 2\cdot\frac{1879}{2520} < \frac{3}{2}$$

Method 2

$$\sum_{k=1}^{\infty} a_k = 2\ln 2 \approx 2(0.69...) \approx 1.38 \quad \text{(recognizing the alternating harmonic series)}$$

Method 3

$\sum_{k=1}^{8} a_k \approx 2(0.634)$. The error stopping at the 8th term satisfies $|\text{error}| \le 2\left(\frac{1}{9}\right)$.

$$1 \le 2(0.634 \pm 0.111) < \frac{3}{2}$$

1970 BC7

The function f is defined for all $x \neq 0$ in the interval $-1 < x < 1$ by $f(x) = \dfrac{x - \sin 2x}{\sin x}$.

(a) How should $f(0)$ be defined in order that f be continuous for all x in the interval $-1 < x < 1$?

(b) With $f(0)$ defined as in part (a), use the definition of the derivative to determine whether $f'(0)$ exists. Show your work.

1970 BC 7
Solution

(a) $f(x)=\dfrac{x-\sin 2x}{\sin x}=\dfrac{x}{\sin x}-2\cos x$, so $\lim\limits_{x\to 0} f(x)=1-2=-1$

Or $f(x)=\dfrac{x-\left(2x-\dfrac{8x^3}{6}+\cdots\right)}{x-\dfrac{x^3}{6}+\cdots}=\dfrac{-1+\dfrac{4}{3}x^2-\dfrac{4}{15}x^4+\cdots}{1-\dfrac{x^2}{6}+\dfrac{x^4}{120}-\cdots}\to 1$ as $x\to 0$.

Or $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{1-2\cos 2x}{\cos x}=-1$

Therefore defining $f(0)=-1$ will make *f* continuous for all *x* in the interval $-1<x<1$.

(b)
$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{h-\sin 2h}{\sin h}+1}{h}$$

$$=\lim_{h\to 0}\frac{h-\sin 2h+\sin h}{h\sin h} \qquad \left[=\frac{0}{0}\right]$$

$$=\lim_{h\to 0}\frac{1-2\cos 2h+\cos h}{h\cos h+\sin h} \qquad \left[=\frac{0}{0}\right]$$

$$=\lim_{h\to 0}\frac{4\sin 2h-\sin h}{-h\sin h+2\cos h}=\frac{0}{2}=0$$

Therefore $f'(0)$ exists and has the value 0.

1971 AB1

Let $f(x) = \ln x$ for all $x > 0$, and let $g(x) = x^2 - 4$ for all real x.
Let H be the composition of f with g, that is $H(x) = f(g(x))$.
Let K be the composition of g with f, that is $K(x) = g(f(x))$.

(a) Find the domain of H.

(b) Find the range of H.

(c) Find the domain of K.

(d) Find the range of K.

(e) Find $H'(7)$.

1971 AB1
Solution

$H(x) = f(g(x)) = \ln(x^2 - 4)$

(a) The domain of H is the set of x for which $x^2 - 4 > 0$, that is, $x > 2$ or $x < -2$. There are other equivalent ways to write this set.

(b) The range of H is the set of all real numbers.

$K(x) = g(f(x)) = (\ln x)^2 - 4$

(c) The domain of K is the same as the domain of the natural logarithm, that is, $x > 0$.

(d) The range of K is the set of all real numbers ≥ -4.

(e) $H'(x) = f'(g(x))g'(x) = \dfrac{1}{g(x)} \cdot 2x = \dfrac{2x}{x^2 - 4}$

$H'(7) = \dfrac{14}{45}$

1971 AB2

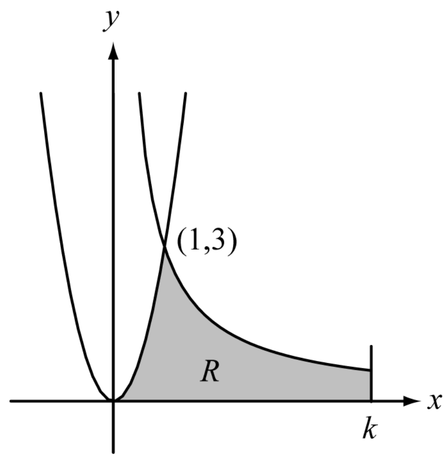
Let R be the region in the first quadrant that lies below both of the curves $y = 3x^2$ and $y = \frac{3}{x}$ and to the left of the line $x = k$ where $k > 1$.

(a) Find the area of R as a function of k.

(b) When the area of R is 7, what is the value of k?

(c) If the area of R is increasing at the constant rate of 5 square units per second, at what rate is k increasing when $k = 15$?

1971 AB2
Solution



(a) Let $A(k)$ be the area of R to the left of the line $x = k$.

$$A(k) = \int_0^1 3x^2\,dx + \int_1^k \frac{3}{x}\,dx = x^3\Big|_0^1 + 3\ln x\Big|_1^k = 1 + 3\ln k$$

(b) $7 = A(k) = 1 + 3\ln k$
$\ln k = 2$
$k = e^2$

(c) $$\frac{dA}{dt} = \frac{dA}{dk}\cdot\frac{dk}{dt} = \frac{3}{k}\frac{dk}{dt}$$

$$5 = \frac{3}{k}\frac{dk}{dt}$$

$$\frac{dk}{dt} = \frac{5k}{3}$$

When $k = 15$, $\dfrac{dk}{dt} = 25$

1971 AB3

Consider $f(x) = \cos^2 x + 2\cos x$ over one complete period beginning with $x = 0$.

(a) Find all values of x in this period at which $f(x) = 0$.

(b) Find all values of x in this period at which the function has a minimum. Justify your answer.

(c) Over what intervals in this period is the curve concave up?

1971 AB3
Solution

$f(x)=\cos^2 x+2\cos x$ in $[0,2\pi)$

(a) $f(x)=\cos x(\cos x+2)=0$ when $\cos x=0$. So $x=\frac{\pi}{2}$ or $x=\frac{3\pi}{2}$.

(b) $f'(x)=-2\cos x\sin x-2\sin x=-2\sin x(1+\cos x)$
$f'(x)=0$ when $x=0$, π (and 2π).

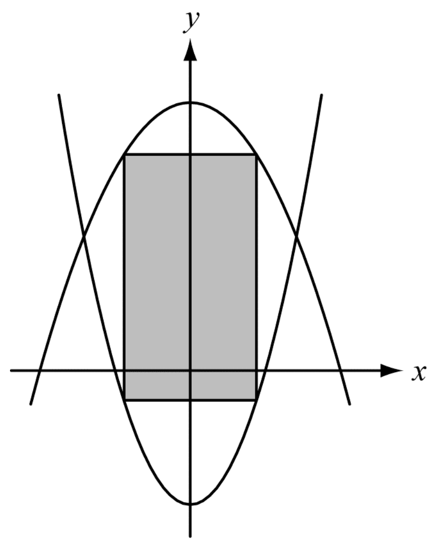
Possible justifications for the location of the minimum:

First derivative test: $f'(x)<0$ for $0<x<\pi$ so the graph is decreasing on this interval. $f'(x)>0$ for $\pi<x<2\pi$ so the graph is increasing on this interval. Therefore there is a minimum at $x=\pi$.	Second derivative test: $f''(x)=-2\cos^2 x+2\sin^2 x-2\cos x$ $=2(1-2\cos x)(1+\cos x)$ $f''(\pi)=0$ so provides no conclusion. But $f''(x)>0$ for x just less than π and just greater than π, thus the graph is concave up in an interval containing $x=\pi$, so $x=\pi$ gives a local minimum. Since it is the only interior critical point, it must be the location of the absolute minimum.
Test the critical points: $f(0)=3$ $f(\pi)=-1$ $f(2\pi)=3$ Therefore there is a minimum at $x=\pi$	Non-calculus reasoning: $f(x)=(\cos x+1)^2-1$ Because of the square, the minimum will occur when $\cos x+1=0$, i.e. when $x=\pi$.

(c) $f''(x)=2(1-2\cos x)(1+\cos x)$
$f''(x)=0$ at $x=\frac{\pi}{3}$ and $x=\frac{5\pi}{3}$.
$f''(x)>0$ for $\frac{\pi}{3}<x<\frac{5\pi}{3}$. Therefore the graph is concave up for $\frac{\pi}{3}<x<\frac{5\pi}{3}$.

1971 AB4/BC1

Find the area of the largest rectangle (with sides parallel to the coordinate axes) that can be inscribed in the region enclosed by the graphs of $f(x) = 18 - x^2$ and $g(x) = 2x^2 - 9$.



1971 AB4/BC1
Solution

Let $A(x)$ be the area of the rectangle with side located at the coordinate x on the x-axis for $x \geq 0$. Then

$A = 2x((18 - x^2) - (2x^2 - 9)) = 2x(27 - 3x^2) = 6(9x - x^3)$ for $0 \leq x \leq 3$.

$A' = 6(9 - 3x^2) = 0$ at $x = \sqrt{3}$

$A'' = 6(-6x) < 0$ at $x = \sqrt{3}$. Therefore there is a local maximum at $x = \sqrt{3}$. But since this is the only critical value for $x > 0$, it must be an absolute maximum.

Alternatively, the absolute maximum must occur at $x = \sqrt{3}$ or at one of the endpoints. But $A = 0$ at $x = 0, 3$. Therefore the absolute maximum is at $x = \sqrt{3}$.

The maximum area is $6(9\sqrt{3} - 3\sqrt{3}) = 36\sqrt{3}$.

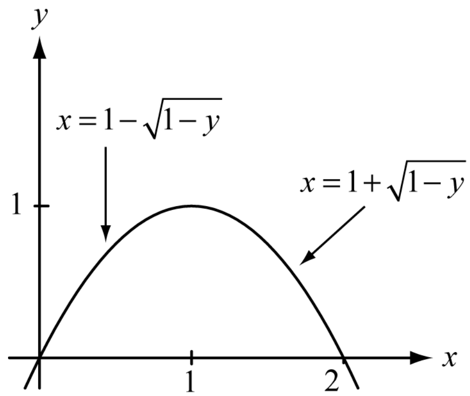
1971 AB5

Let R be the region in the first quadrant bounded by the x-axis and the curve $y = 2x - x^2$.

(a) Find the volume produced when R is revolved about the x-axis.

(b) Find the volume produced when R is revolved about the y-axis.

1971 AB5
Solution



(a) $\text{Volume} = \pi\int_0^2 (2x-x^2)^2\,dx = \pi\int_0^2 (4x^2-4x^3+x^4)\,dx$

$$= \pi\left(\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right)\Bigg|_0^2 = \frac{16}{15}\pi$$

(b) Shells:

$\text{Volume} = 2\pi\int_0^2 xy\,dx = 2\pi\int_0^2 x(2x-x^2)\,dx = 2\pi\int_0^2 (2x^2-x^3)\,dx$

$$= 2\pi\left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_0^2 = 2\pi\left(\left(\frac{16}{3}-\frac{16}{4}\right)-0\right) = \frac{8\pi}{3}$$

Disks:

$\text{Volume} = \pi\int_0^1 ((1+\sqrt{1-y})^2 - (1-\sqrt{1-y})^2)\,dy$

$$= \pi\int_0^1 ((1+2\sqrt{1-y}+1-y) - (1-2\sqrt{1-y}+1-y))\,dy$$

$$= \pi\int_0^1 4\sqrt{1-y}\,dy$$

$$= -4\pi\left(\frac{2}{3}\right)(1-y)^{3/2}\Bigg|_0^1 = -\frac{8\pi}{3}(0-1) = \frac{8\pi}{3}$$

1971 AB6

A particle starts at the point $(5,0)$ at $t=0$ and moves along the x-axis in such a way that at time $t>0$ its velocity $v(t)$ is given by $v(t)=\frac{t}{1+t^2}$.

(a) Determine the maximum velocity attained by the particle. Justify your answer.

(b) Determine the position of the particle at $t=6$.

(c) Find the limiting value of the velocity as t increases without bound.

(d) Does the particle ever pass the point $(500,0)$? Explain.

1971 AB6
Solution

(a) $v'(t)=\dfrac{1-t^2}{(1+t^2)^2}=0$ at $t=1$.

$v'(t)>0$ for $0<t<1$ and $v'(t)>0$ for $t>1$. Therefore the velocity is a maximum at $t = 1$ and the maximum velocity is $v(1)=\dfrac{1}{2}$.

Alternatively, $v''(t)=\dfrac{2t(t^2-3)}{(1+t^2)^3}$ and so $v''(1)=-\dfrac{1}{2}<0$. Therefore v has a local maximum at $t = 1$, but since this is the only critical point for $t>0$, it must also be an absolute maximum. Therefore the maximum velocity is $v(1)=\dfrac{1}{2}$.

(One could also justify that since $v(t)=\dfrac{1}{t+\frac{1}{t}}$ and $t+\dfrac{1}{t}$ has a minimum value of 2, then $v(t)$ has a maximum value of $\dfrac{1}{2}$.)

(b) $s(t)=\int v(t)dt=\dfrac{1}{2}\ln(1+t^2)+C$

$5=s(0)=\dfrac{1}{2}\ln(1)+C=C$

Hence $s(t)=5+\dfrac{1}{2}\ln(1+t^2)$ and so $s(6)=5+\dfrac{1}{2}\ln(37)=5+\ln\sqrt{37}$.

(c) $\lim\limits_{t\to\infty} v(t)=\lim\limits_{t\to\infty}\dfrac{t}{1+t^2}=\lim\limits_{t\to\infty}\dfrac{\frac{1}{t}}{\frac{1}{t^2}+1}=0$

or

$\lim\limits_{t\to\infty} v(t)=\lim\limits_{t\to\infty}\dfrac{t}{1+t^2}=\lim\limits_{t\to\infty}\dfrac{1}{2t}=0$ using L’Hôpital’s rule.

(d) Yes, the particle does pass the point (500, 0) since the natural logarithm is an unbounded function.

$500\le 5+\dfrac{1}{2}\ln(1+t^2)$ for $t\ge\sqrt{e^{990}-1}$

1971 AB7/BC3

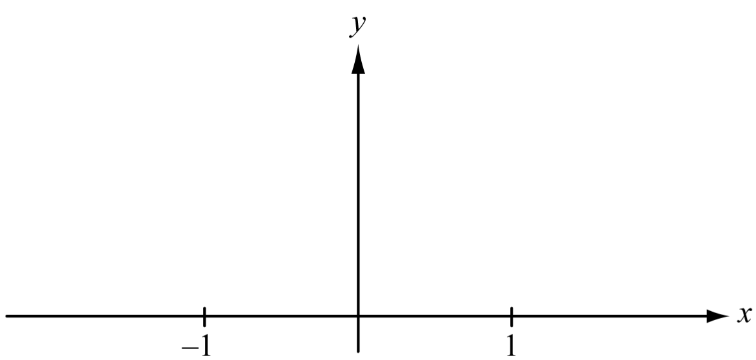
Let f be the function defined by $f(x)=|x|^{1/2}e^{-x^2}$ for all real numbers x.

(a) Describe the symmetry of the graph of f.

(b) Over what intervals of the domain is this function increasing?

(c) Sketch the graph of f on the axes provided showing clearly:

- (i) behavior near the origin,
- (ii) maximum and minimum points,
- (iii) behavior for large $|x|$.



1971 AB7/BC3
Solution

(a) The graph is symmetric with respect to the y-axis since

$$f(-x) = |-x|^{1/2} e^{-(-x)^2} = |x|^{1/2} e^{-x^2} = f(x)$$

(b) For $x > 0$,

$$f'(x) = \frac{1}{2}x^{-1/2}e^{-x^2} - x^{1/2}2xe^{-x^2} = \frac{1}{2}x^{-1/2}e^{-x^2}(1-4x^2) = \frac{1}{2}x^{-1/2}e^{-x^2}(1-2x)(1+2x)$$

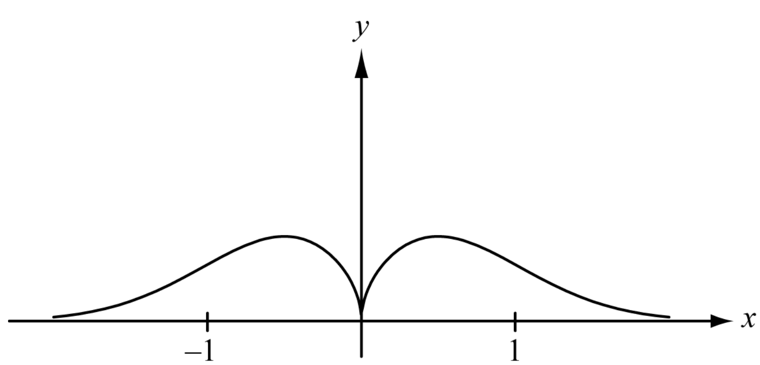
Therefore the function is increasing on the intervals $\left(-\infty, -\frac{1}{2}\right)$ and $\left(0, \frac{1}{2}\right)$.

(c) (i) there is a vertical tangent at (0, 0)

(ii) the maximum points are at $x = \pm\frac{1}{2}$, where $y = \frac{1}{\sqrt{2}e^{1/4}} \approx 0.55$.

The minimum point is (0, 0).

(iii) $\lim_{x\to\infty} f(x) = 0$



1971 BC2

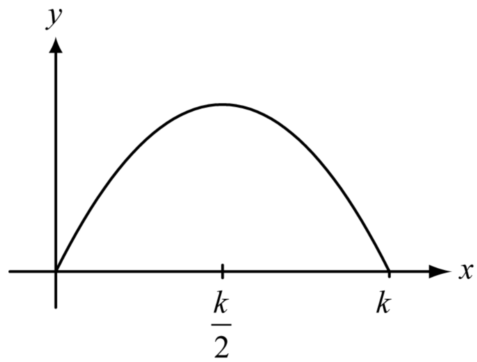
Let R be the region of the first quadrant bounded by the x-axis and the curve $y = kx - x^2$, where $k > 0$.

(a) In terms of k, find the volume produced when R is revolved about the x-axis.

(b) In terms of k, find the volume produced when R is revolved about the y-axis.

(c) Find the value of k for which the volumes found in parts (a) and (b) are equal.

1971 BC2
Solution



(a) $\text{Volume} = \pi\int_0^k (kx - x^2)^2\,dx = \pi\int_0^k (k^2x^2 - 2kx^3 + x^4)dx$

$$= \pi\left(\frac{k^2x^3}{3} - \frac{kx^4}{2} + \frac{x^5}{5}\right)\Bigg|_0^k = \frac{\pi k^5}{30}$$

(b) $\text{Volume} = 2\pi\int_0^k x(kx - x^2)\,dx = 2\pi\int_0^k (kx^2 - x^3)\,dx = 2\pi\left(\frac{kx^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^k = \frac{\pi k^4}{6}$

or

$$\text{Volume} = \pi\int_0^{\frac{k^2}{4}} \left(\left(\frac{k + \sqrt{k^2 - 4y}}{2}\right)^2 - \pi\left(\frac{k - \sqrt{k^2 - 4y}}{2}\right)^2\right)dy$$

$$= \pi\int_0^{\frac{k^2}{4}} \left(k\sqrt{k^2 - 4y}\right)dy = \pi\frac{-k}{6}(k^2 - 4y)^{3/2}\Bigg|_0^{\frac{k^2}{4}} = \frac{\pi k^4}{6}$$

(c) $\frac{\pi k^5}{30} = \frac{\pi k^4}{6}$ when $k = 5$

1971 BC4

(a) Write the first three nonzero terms in the Taylor series expansion of $\cos x$ about $x = \frac{\pi}{2}$.

(b) What is the interval of convergence of the Taylor series mentioned in part (a)? Show your method.

(c) Use the first two nonzero terms of the series in part (a) to approximate $\cos\left(\frac{\pi}{2} + 0.1\right)$.

(d) Estimate the accuracy of the approximation found in part (c). Show your method.

1971 BC4
Solution

(a)

$$
\begin{aligned}
f(x) &= \cos x & f(\pi/2) &= 0 \\
f'(x) &= -\sin x & f'(\pi/2) &= -1 \\
f''(x) &= -\cos x & f''(\pi/2) &= 0 \\
f'''(x) &= \sin x & f'''(\pi/2) &= 1 \\
f^{(4)}(x) &= \cos x & f^{(4)}(\pi/2) &= 0 \\
f^{(5)}(x) &= -\sin x & f^{(5)}(\pi/2) &= -1
\end{aligned}
$$

$$f(x) = \cos x = -\left(x-\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} - \frac{\left(x-\frac{\pi}{2}\right)^5}{5!} + \cdots$$

(b) $$R = \left| -\frac{\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{\left(x-\frac{\pi}{2}\right)^{2n-1}} \right| = \frac{\left(x-\frac{\pi}{2}\right)^2}{(2n+1)(2n)}$$

$\lim\limits_{n\to\infty} R = 0 < 1$ for all x and so the series converges for all x.

or

For all x, the series is strictly alternating with $\lim\limits_{n\to\infty} u_n \to 0$ and $|u_{n+1}| < |u_n|$ for all n sufficiently large. The series therefore converges for all x by the Alternating Series Test.

(c) $$\cos\left(\frac{\pi}{2}+0.1\right) \approx (0.1) + \frac{(0.1)^3}{3!} = -0.1 + \frac{0.001}{6} = -\frac{0.599}{6} \approx -0.09983$$

(d) Since this is an alternating series with the terms decreasing to 0 in absolute value starting with the first term, the error satisfies $|\text{error}| < \frac{(0.1)^5}{5!} = 8.\bar{3}\times 10^{-8}$.

or

Could use remainder form to get $|\text{error}| < \frac{(0.1)^4}{4!} = 4.1\bar{6}\times 10^{-6}$

1971 BC5

Determine whether or not $\int_0^{\infty} xe^{-x}dx$ converges. If it converges, give its value. Show your reasoning.

1971 BC5
Solution

Let $I=\int xe^{-x}\,dx$. Use integration by parts with

$$u=x \qquad dv=e^{-x}dx$$
$$du=dx \quad v=-e^{-x}$$

$$I=-xe^{-x}+\int e^{-x}\,dx=e^{-x}(-x-1)$$

$$\begin{aligned}\int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x}\,dx\\ &= \lim_{b\to\infty}(-xe^{-x}-e^{-x})\Big|_0^b\\ &= \lim_{b\to\infty}(-be^{-b}-e^{-b}+1)\\ &= \lim_{b\to\infty}\left(-\frac{b}{e^b}-\frac{1}{e^b}+1\right)\\ &=1\end{aligned}$$

Therefore the integral converges and has the value 1.

Note: $\int_0^\infty xe^{-x}dx=\Gamma(2)=1$ where $\Gamma(x)$ is the Gamma function. Knowledge of the gamma function could also be used to show that this improper integral converges and has the value 1.

1971 BC6

Find a function f that has a continuous derivative on $(0, \infty)$ and that has both of the following properties:

(i) The graph of f goes through the point $(1,1)$.
(ii) The length L of the curve from $(1,1)$ to any point $(x, f(x))$ is given by $L = \ln x + f(x) - 1$.

<u>Note:</u>

Recall that the arc length from $(a, f(a))$ to $(b, f(b))$ is given by $\int_a^b \sqrt{1+(f'(t))^2}\,dt$.

1971 BC6
Solution

$$L = \ln x + f(x) - 1 = \int_1^x \sqrt{1 + (f'(t))^2}\, dt$$

Differentiate both sides

$$\frac{1}{x} + f'(x) = \sqrt{1 + (f'(x))^2}$$

Square and solve for $f'(x)$

$$\frac{1}{x^2} + \frac{2f'(x)}{x} + (f'(x))^2 = 1 + (f'(x))^2$$

$$f'(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

Antidifferentiate

$$f(x) = \frac{1}{2}\left(\frac{x^2}{2} - \ln x\right) + C$$

$$1 = f(1) = \frac{1}{2}\left(\frac{1}{2} - 0\right) + C = \frac{1}{4} + C$$

$$C = \frac{3}{4}$$

Therefore $f(x) = \dfrac{x^2}{4} - \dfrac{1}{2}\ln x + \dfrac{3}{4}$

1971 BC7

Definition: A function $f,$ defined for all positive numbers, is "tractible" if and only if for every $\varepsilon > 0$, there exists an integer $M > 0$ such that $x > M$ implies that $\left| f(x) - x \right| < \varepsilon$.

(a) If $h(x) = x - 3$ for all positive real numbers $x,$ prove that h is not tractible.

(b) If $g(x) = x + \dfrac{1}{x^2}$ for all positive real numbers $x,$ prove that g is tractible.

(c) Discuss the graphical significance of tractibility.

1971 BC7
Solution

(a) $|h(x)-x| = |(x+3)-x| = 3$. It is not possible to make $3 < \varepsilon$ for every $\varepsilon > 0$. Therefore $h(x) = x - 3$ is not tractible.

(b) $|g(x)-x| = \left| x + \frac{1}{x^2} - x \right| = \frac{1}{x^2} < \varepsilon$ for $x > \sqrt{\frac{1}{\varepsilon}}$. Choose $M = \left\lfloor \sqrt{\frac{1}{\varepsilon}} \right\rfloor + 1$ (where $\lfloor x \rfloor$ is the greatest integer function), or choose M to be any integer greater than $\sqrt{\frac{1}{\varepsilon}}$. This choice of M for every $\varepsilon > 0$ will satisfy the condition for $g(x)$ to be tractible.

(c) As x increases without bound, the vertical difference between a tractible function and the line $y = x$ will approach zero. This means that the graph of a tractible function approaches the line $y = x$ asymptotically.

1972 AB1

Let $f(x) = 4x^3 - 3x - 1$.

(a) Find the x-intercepts of the graph of f.

(b) Write an equation for the tangent line to the graph of f at $x = 2$.

(c) Write an equation of the graph that is the reflection across the y-axis of the graph of f.

1972 AB1
Solution

(a) Must solve $f(x) = 4x^3 - 3x - 1 = 0$. The polynomial can be factored by using long division or synthetic division. Possible rational zeros are $\pm 1, \pm\frac{1}{4}, \pm\frac{1}{2}$. Since $f(1) = 4 - 3 - 1 = 0$, divide the polynomial by $x - 1$.

$$\begin{array}{r|l} & 4x^2 + 4x + 1 \\ \hline x-1 & 4x^3 \qquad\quad - 3x - 1 \\ & \underline{4x^3 - 4x^2} \\ & \qquad +4x^2 - 3x \\ & \qquad \underline{+4x^2 - 4x} \\ & \qquad\qquad\quad x - 1 \end{array}$$

$$\begin{array}{c|cccc} 1 & 4 & 0 & -3 & -1 \\ & & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

Therefore $f(x) = (x-1)(4x^2 + 4x + 1) = (x-1)(2x+1)^2$ and the x-intercepts are $x = 1$ and $x = -\frac{1}{2}$.

(b) $f'(x) = 12x^2 - 3$
$f'(2) = 48 - 3 = 45$
$f(2) = 25$
The equation of the tangent line is $y - 25 = 45(x - 2)$ or $y = 45x - 65$.

(c) The reflection across the y-axis is given by $f(-x)$. So the equation is
$y = f(-x) = 4(-x)^3 - 3(-x) - 1 = -4x^3 + 3x - 1$

1972 AB2/BC1

A particle starts at time $t = 0$ and moves on a number line so that its position at time t is given by $x(t) = (t-2)^3(t-6)$.

(a) When is the particle moving to the right?

(b) When is the particle at rest?

(c) When does the particle change direction?

(d) What is the farthest to the left of the origin that the particle moves?

1972 AB2/BC1
Solution

$$x(t) = (t-2)^3(t-6)$$

$$v(t) = x'(t) = (t-2)^3 + 3(t-2)^2(t-6) = (t-2)^2((t-2)+3(t-6))$$
$$= (t-2)^2(4t-20) = 4(t-2)^2(t-5)$$

$$a(t) = x''(t) = 12(t-2)(t-4)$$

(a) The particle is moving to the right when $v(t) = 4(t-2)^2(t-5) > 0$. This happens for $t > 5$.

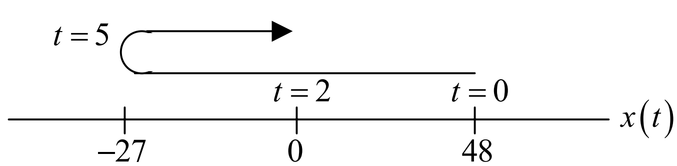
(b) The particle is at rest when $v(t) = 4(t-2)^2(t-5) = 0$. This happens at $t = 2$ and $t = 5$.

(c) The particle changes direction when the velocity changes sign. The velocity is negative just to the left and just to the right of $t = 2$. The velocity is negative for $t < 5$ and positive for $t > 5$. Therefore the particle only changes direction at $t = 5$.

(d) The minimum value of x is at $t = 5$ since the particle moves to the left for $t < 5$ and moves back to the right for $t > 5$.

$$x(5) = 3^3(-1) = -27$$

Therefore the farthest to the left of the origin that the particle moves is $x = -27$.



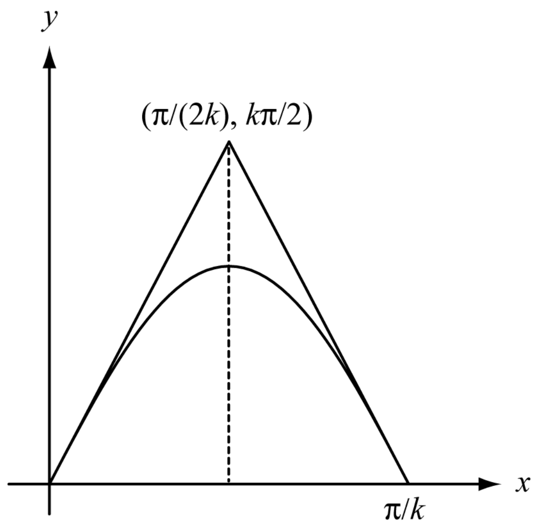
1972 AB3/BC2

Let $f(x) = k\sin(kx)$ where k is a positive constant.

(a) Find the area of the region bounded by one arch of the graph of f and the x-axis.

(b) Find the area of the triangle formed by the x-axis and the tangents to one arch of f at the points where the graph of f crosses the x-axis.

1972 AB3/BC2
Solution



(a) Area $= \int_0^{\frac{\pi}{k}} k\sin(kx)\,dx = -\cos(kx)\Big|_0^{\pi/k} = -\cos\pi - (-\cos 0) = 1 + 1 = 2$

(b) $f'(x) = k^2\cos(kx)$

$f'(0) = k^2$ and so the equation of the left tangent line is $y = k^2 x$.

$f'\left(\frac{\pi}{k}\right) = -k^2$ and so the equation of the right tangent line is $y = -k^2\left(x - \frac{\pi}{k}\right)$.

The tangent lines intersect when

$k^2 x = -k^2\left(x - \frac{\pi}{k}\right)$

$2k^2 x = k\pi$

$x = \frac{\pi}{2k}$

At this x value, $y = k^2\left(\frac{\pi}{2k}\right) = \frac{k\pi}{2}$.

The area of the triangle is therefore $\frac{1}{2}\left(\frac{\pi}{k}\right)\left(\frac{k\pi}{2}\right) = \frac{\pi^2}{4}$.

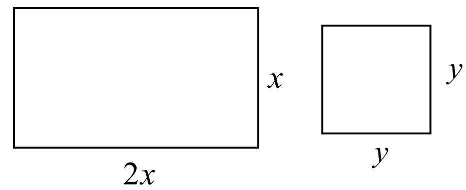
1972 AB4/BC3

A man has 340 yard of fencing for enclosing two separate fields, one of which is to be a rectangle twice as long as it is wide and the other a square. The square field must contain at least 100 square yards and the rectangular one must contain at least 800 square yards.

(a) If x is the width of the rectangular field, what are the maximum and minimum possible values of x?

(b) What is the greatest number of square yards that can be enclosed in the two fields? Justify your answer.

1972 AB4/ BC3
Solution



(a) Let y be the width of the square field. Then $6x + 4y = 340$.

Since $y^2 \geq 100$, then $y \geq 10$.
Since $2x^2 \geq 800$, then $x \geq 20$.

$$4y = 340 - 6x \Rightarrow y = 85 - \frac{3}{2}x$$
$$y \geq 10 \Rightarrow 85 - \frac{3}{2}x \geq 10 \Rightarrow -\frac{3}{2}x \geq -75 \Rightarrow x \leq 50$$

Therefore the maximum value of x is 50; the minimum value of x is 20.

(b) Let A be the total area. Then

$$A = 2x^2 + y^2 = 2x^2 + \left(85 - \frac{3}{2}x\right)^2$$
$$A' = 4x + 2\left(85 - \frac{3}{2}x\right)\left(-\frac{3}{2}\right) = 4x - 255 + \frac{9}{2}x = \frac{17}{2}x - 255$$
$A' = 0$ only when $x = \frac{2}{17} \cdot 255 = 30$.

But $A'' = \frac{17}{2} > 0$ and so $x = 30$ gives a relative minimum for A. Hence the maximum area must occur at $x = 20$ or at $x = 50$.

At $x = 20$, $A = 2(20)^2 + (55)^2 = 3825$
At $x = 50$, $A = 2(50)^2 + (10)^2 = 5100$

Therefore, 5100 is the greatest number of square yards that can be enclosed.

1972 AB5

Let $y = 2e^{\cos x}$.

(a) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) If x and y both vary with time in such a way that y increases at a steady rate of 5 units per second, at what rate is x changing when $x = \frac{\pi}{2}$?

1972 AB5
Solution

(a) $y = 2e^{\cos x}$

$$\frac{dy}{dx} = 2e^{\cos x}(-\sin x) = -2(\sin x)e^{\cos x}$$

$$\frac{d^2y}{dx^2} = -2(\sin x)e^{\cos x}(-\sin x) - 2(\cos x)e^{\cos x} = 2e^{\cos x}(\sin^2 x - \cos x)$$

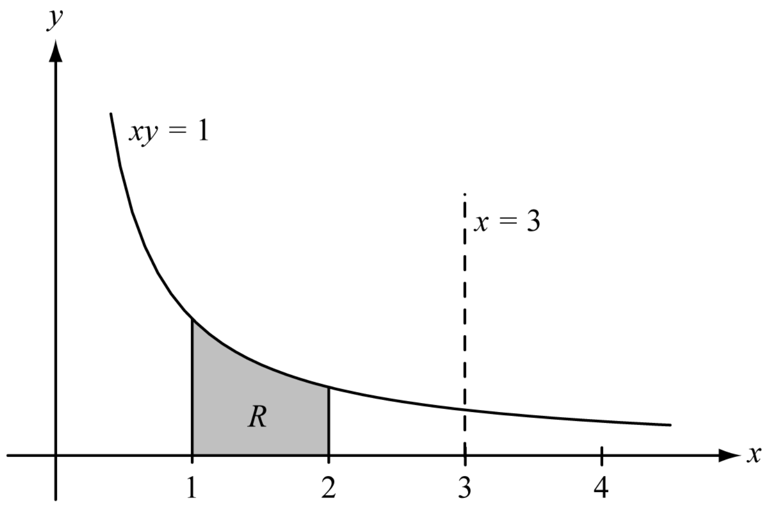
(b) $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = -2(\sin x)e^{\cos x} \cdot \frac{dx}{dt}$$

Substituting $\frac{dy}{dt} = 5$ and $x = \frac{\pi}{2}$ gives

$$5 = -2\left(\sin\frac{\pi}{2}\right)e^{\cos(\pi/2)}\frac{dx}{dt} = -2(1)e^{0}\frac{dx}{dt} = -2\frac{dx}{dt}.$$

Therefore $\frac{dx}{dt} = -\frac{5}{2}$ when $x = \frac{\pi}{2}$.

1972 AB6



In the figure, the shaded region R is bounded by the graphs of $xy = 1$, $x = 1$, $x = 2$, and $y = 0$.

(a) Find the volume of the solid generated by revolving the region R about the x-axis.

(b) Find the volume of the solid generated by revolving the region R about the line $x = 3$.

1972 AB6
Solution

(a) $\text{Volume} = \pi\int_1^2 y^2\,dx = \pi\int_1^2 \left(\frac{1}{x}\right)^2 dx = -\pi\frac{1}{x}\Big|_1^2 = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$

(b) $\text{Volume} = 2\pi\int_1^2 (3-x)y\,dx = 2\pi\int_1^2 (3-x)\frac{1}{x}\,dx = 2\pi\int_1^2 \left(\frac{3}{x} - 1\right)dx$

$= 2\pi(3\ln x - x)\Big|_1^2 = 2\pi((3\ln 2 - 2) - (-1)) = 2\pi(3\ln 2 - 1)$

1972 AB7

A function f is defined for all real numbers and has the following three properties:

(i) $f(1) = 5$,
(ii) $f(3) = 21$, and
(iii) for all real values of a and b, $f(a+b) - f(a) = kab + 2b^2$ where k is a fixed real number independent of a and b.

(a) Use $a = 1$ and $b = 2$ to find the value of k.

(b) Find $f'(3)$.

(c) Find $f'(x)$ and $f(x)$ for all real x.

1972 AB7
Solution

(a) Let a = 1 and b = 2 in (iii):

$$f(1+2)-f(1)=k(1)(2)+2(2)^2$$
$$f(3)-f(1)=2k+8$$

From (i) and (ii),
$21-5=2k+8$

Hence k = 4.

(b) Method 1 (using (iii))

$$f'(3)=\lim_{h\to0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to0}\frac{4\cdot3\cdot h+2h^2}{h}=\lim_{h\to0}(12+2h)=12$$

Method 2 (using (iii))

$$f'(3)=\lim_{t\to3}\frac{f(3)-f(t)}{3-t}=\lim_{t\to3}\frac{f(t+(3-t))-f(t)}{3-t}=\lim_{t\to3}\frac{4t(3-t)+2(3-t)^2}{3-t}$$
$$=\lim_{t\to3}(4t+2(3-t))=12$$

(c) $$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{4xh+2h^2}{h}=\lim_{h\to0}(4x+2h)=4x$$

$$f(x)=\int 4x\,dx=2x^2+C$$
$5=f(1)=2(1)^2+C=2+C$, so C = 3. Therefore $f(x)=2x^2+3$.

Alternate method:

Let $a=1$, $b=x-1$, and $k=4$ in (iii).

$$f(1+(x-1))-f(1)=4(x-1)+2(x-1)^2=2x^2-2$$

Since $f(1)=5$, this gives $f(x)=2x^2+3$ and so $f'(x)=4x$.

1972 BC4

(a) Does the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{3k+5}}$ converge? Justify your answer.

(b) Determine all values of x for which the series $\sum_{k=1}^{\infty} \frac{2^k x^k}{k}$ converges. Justify your answer.

1972 BC4
Solution

(a) The terms of the series satisfy the conditions needed to use the integral test.

$$\int_1^\infty \frac{1}{\sqrt{3x+5}}\,dx = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{3x+5}}\,dx = \lim_{b\to\infty}\int_1^b (3x+5)^{-1/2}\,dx$$

$$= \lim_{b\to\infty}\frac{2}{3}(3x+5)^{1/2}\Big|_1^b = \lim_{b\to\infty}\left(\frac{2}{3}(3b+5)^{1/2}-\frac{2}{3}(8)^{1/2}\right)=\infty$$

Since the improper integral diverges, so does the series.

(b) Ratio Test

$$\lim_{k\to\infty}\left|\frac{\dfrac{2^{k+1}x^{k+1}}{k+1}}{\dfrac{2^k x^k}{k}}\right| = \lim_{k\to\infty}\left|\frac{2^{k+1}x^{k+1}}{k+1}\cdot\frac{k}{2^k x^k}\right| = \lim_{k\to\infty}\left|2x\cdot\frac{k}{k+1}\right| = |2x|$$

The series converges for $|2x|<1$ or $-\frac{1}{2}<x<\frac{1}{2}$.

At $x=\frac{1}{2}$, the series is $\sum_{k=1}^{\infty}\frac{2^k\left(\frac{1}{2}\right)^k}{k} = \sum_{k=1}^{\infty}\frac{1}{k}$ which diverges since this is the harmonic series.

At $x=-\frac{1}{2}$, the series is $\sum_{k=1}^{\infty}\frac{2^k\left(-\frac{1}{2}\right)^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ which converges since this is the alternating harmonic series.

The interval of convergence is $-\frac{1}{2}\le x<\frac{1}{2}$.

1972 BC5

(a) Determine the general solution of the differential equation $y'' - 4y' + 4y = e^{2x}$.

(b) Assume that xe^{x} is a solution of the differential equation $y'' + ay' + by = 0$. Determine the constants a and b.

1972 BC5
Solution

(a) First find the solution to associated homogeneous equation $y'' - 4y' + 4 = 0$.

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$

Therefore the homogeneous solution is $y_h = C_1 e^{2x} + C_2 x e^{2x}$.

Find a particular solution using the method of undetermined coefficients.

$y_p = Ax^2 e^{2x}$

$y_p' = 2Ax^2 e^{2x} + 2Axe^{2x} = 2Ae^{2x}(x^2 + x)$

$y_p'' = 2Ae^{2x}(2x+1) + 4Ae^{2x}(x^2 + x) = 2Ae^{2x}(2x^2 + 4x + 1)$

Substituting into $y'' - 4y' + 4y = e^{2x}$ gives $2Ae^{2x} = e^{2x}$ and so $A = \frac{1}{2}$.

Therefore $y_p = \frac{1}{2}x^2 e^{2x}$.

The general solution is $y = y_h + y_p = C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{2}x^2 e^{2x}$

(b) $y = xe^x$

$y' = e^x(x+1)$

$y'' = e^x(x+2)$

Substitute into $y'' + ay' + by = 0$

$e^x(x+2) + ae^x(x+1) + bxe^x = 0$

$(2+a)e^x + (1+a+b)xe^x = 0$

$\left.\begin{array}{l} 2 + a = 0 \\ 1 + a + b = 0 \end{array}\right\}$ (since the equality must hold for all x)

$a = -2$

$b = 1$

1972 BC6

Consider the function f defined by $f(x) = \begin{cases} \dfrac{x}{\ln x} & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ \dfrac{-x}{\ln(-x)} & \text{if } x < 0 \end{cases}$

(a) For what values of x is f continuous?

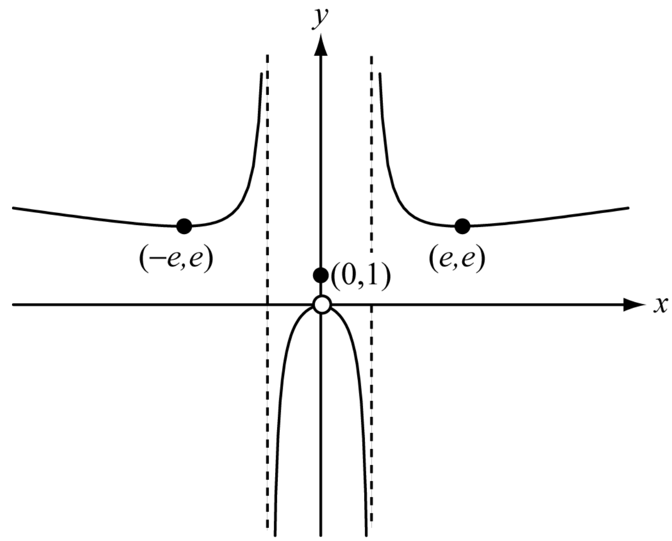
(b) Is the graph of f symmetric with respect to

(i) the y-axis?
(ii) the origin?

(c) Find the coordinates of all relative maximum points.

(d) Find the coordinates of all relative minimum points.

1972 BC6
Solution



(a) $\lim_{x\to 0^+} \frac{x}{\ln x} = 0 \neq f(0) = 1$

Therefore f is not continuous at $x = 0$.
f is continuous at all other values of x except for $x = 1$ and $x = -1$ where the function is not defined.

(b) For $x > 0$,

$$f(-x) = \frac{-(-x)}{\ln(-(-x))} = \frac{x}{\ln x} = f(x)$$

For $x < 0$,

$$f(-x) = \frac{(-x)}{\ln(-x)} = f(x)$$

(i) The graph is symmetric with respect to the y-axis since $f(-x) = f(x)$ for all x.
(ii) The graph is not symmetric with respect to the origin since $f(-x) \neq -f(x)$.

(c) and (d)

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2} \text{ for } x > 0$$

$$f'(x) = \frac{-\ln(-x) + 1}{(\ln(-x))^2} \text{ for } x < 0$$

$$f''(x) = \frac{\frac{1}{x}(\ln x)^2 - (\ln x - 1)\left(\frac{2}{x}\right)\ln x}{(\ln x)^4} \text{ for } x > 0 .$$

$f(x) < 0$ for $-1 < x < 0$ and for $0 < x < 1$. Since $f(0) = 1$, the point (0,1) is a relative maximum.

For $x > 0$, $f'(x) = 0$ when $\ln x - 1 = 0$, that is, at $x = e$.
$f''(e) = \frac{1}{e} > 0$. Therefore there is a relative minimum at (e,e). By symmetry, there is also a relative minimum at $(-e,e)$.

1972 BC7

Let C be the curve defined from $t = 0$ to $t = 6$ by the parametric equations $x = \frac{t+2}{2}$, $y = t(6-t)$. Let R be the region bounded by C and the x-axis. Set up but do not evaluate an integral expression in terms of a single variable for

(a) the length of C.

(b) the volume of the solid generated by revolving R about the x-axis.

(c) the surface area of the solid generated by revolving R about the x-axis.

1972 BC7
Solution

$$\frac{dx}{dt}=\frac{1}{2},\quad \frac{dy}{dt}=6-2t$$

$y=-4(x^2-5x+4)$ for $1\le x\le 4$, so $\frac{dy}{dx}=-4(2x-5)$

$x=\frac{5}{2}-\frac{1}{2}\sqrt{9-y}$ for $0\le y\le 9$ (left half of curve), so $\frac{dx}{dy}=\frac{1}{4\sqrt{9-y}}$

Each of the integrals can be written in terms of *dt*, *dx*, or *dy*.

(a) Length $=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$$=\int_0^6\sqrt{\frac{1}{4}+(6-2t)^2}\,dt$$

$$=\int_1^4\sqrt{1+16(2x-5)^2}\,dx$$

$$=2\int_0^9\sqrt{1+\frac{1}{16(9-y)}}\,dy$$

(b) Volume $=\pi\int_a^b y^2\,dx$

$$=\pi\int_0^6(t(6-t))^2\cdot\frac{1}{2}dt$$

$$=\pi\int_1^4 16(x^2-5x+4)^2\,dx$$

$$=2\pi\int_0^9 y\sqrt{9-y}\,dy$$

(c) Surface area $=2\pi\int_a^b y\,ds$

$$=2\pi\int_0^6 t(6-t)\sqrt{\frac{1}{4}+(6-2t)^2}\,dt$$

$$=2\pi\int_1^4 -4(x^2-5x+4)\sqrt{1+16(2x-5)^2}\,dx$$

$$=4\pi\int_0^9 y\sqrt{1+\frac{1}{16(9-y)}}\,dy$$

1973 AB1

Given $f(x) = x^3 - 6x^2 + 9x$ and $g(x) = 4$.

(a) Find the coordinates of the points common to the graphs of f and g.

(b) Find all the zeros of f.

(c) If the domain of f is limited to the closed interval $[0, 2]$, what is the range of f? Show your reasoning.

1973 AB1
Solution

(a) $x^3 - 6x^2 + 9x = 4$

$x^3 - 6x^2 + 9x - 4 = 0$

$(x-1)(x^2 - 5x + 4) = 0$

$(x-1)(x-1)(x-4) = 0$

1	1	–6	9	–4
		1	–5	4
	1	–5	4	0

The roots are $x = 1$ and $x = 4$. The coordinates of the common points are therefore $(1,4)$ and $(4,4)$.

(b) $x^3 - 6x^2 + 9x = 0$

$x(x^2 - 6x + 9) = 0$

$x(x-3)^2 = 0$

The zeros of f are $x = 0$ and $x = 3$.

(c) $f'(x) = 3x^2 - 12x + 9 = 3(x-3)(x-1)$
On the interval $[0,2]$, the only critical point is at $x = 1$.
$f''(x) = 6x - 12$ so $f''(1) = -6 < 0$. Therefore there is a relative maximum at $x = 1$.
Alternatively, the graph of f is increasing on the interval (0,1) and decreasing on the interval (1,2) since $f'(x) > 0$ on (0,1) and $f'(x) < 0$ on (1,2). Hence there is an absolute maximum at $x = 1$ on the interval $[0,2]$.

$f(0) = 0$
$f(1) = 4$
$f(2) = 2$

Since there are no other critical points in the restricted domain, the range is the closed interval $[0,4]$.

1973 AB2

A particle moves on the x-axis so that its acceleration at any time $t > 0$ is given by $a = \frac{t}{8} - \frac{1}{t^2}$. When $t = 1$, $v = \frac{9}{16}$, and $s = \frac{25}{48}$.

(a) Find the velocity v in terms of t.

(b) Does the numerical value of the velocity ever exceed 50? Explain.

(c) Find the distance s from the origin at time $t = 2$.

1973 AB2
Solution

(a) $v = \int \left(\frac{t}{8} - \frac{1}{t^2} \right) dt = \frac{t^2}{16} + \frac{1}{t} + C$

At $t = 1$, $\frac{9}{16} = \frac{1}{16} + 1 + C$ and so $C = -\frac{1}{2}$. Therefore $v = \frac{t^2}{16} + \frac{1}{t} - \frac{1}{2}$.

(b) Yes, the numerical value does exceed 50. This can be explained by any one of the following observations:

(i) $\lim_{t \to +\infty} v(t) = \infty$

(ii) $\lim_{t \to 0^+} v(t) = \infty$

(iii) When $t = 32$, for example, $v = \frac{32^2}{16} + \frac{1}{32} - \frac{1}{2} = 64 + \frac{1}{32} - \frac{1}{2} > 50$

(c) $s = \int \left(\frac{t^2}{16} + \frac{1}{t} - \frac{1}{2} \right) dt = \frac{t^3}{48} + \ln t - \frac{1}{2} t + C$

At $t = 1$, $\frac{25}{48} = \frac{1}{48} - \frac{1}{2} + C$ and so $C = 1$. Therefore $s = \frac{t^3}{48} + \ln t - \frac{1}{2} t + 1$.

At $t = 2$, $s = \frac{8}{48} + \ln 2 - 1 + 1 = \frac{1}{6} + \ln 2$.

1973 AB3/BC1

Given the curve $x + xy + 2y^2 = 6$.

(a) Find an expression for the slope of the curve at any point (x, y) on the curve.

(b) Write an equation for the line tangent to the curve at the point $(2,1)$.

(c) Find the coordinates of all other points on this curve with slope equal to the slope at $(2,1)$.

1973 AB3/BC1
Solution

(a) Implicit differentiation gives

$x + xy + 2y^2 = 6$

$1 + x\dfrac{dy}{dx} + y + 4y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{1+y}{x+4y}$

(b) At the point (2,1), $\dfrac{dy}{dx} = -\dfrac{1+1}{2+4} = -\dfrac{1}{3}$. Therefore the equation of the tangent line is

$y - 1 = -\dfrac{1}{3}(x-2)$, or $y = -\dfrac{1}{3}x + \dfrac{5}{3}$, or $3y + x - 5 = 0$.

(c) $-\dfrac{1+y}{x+4y} = -\dfrac{1}{3}$

$-1 - y = -\dfrac{x}{3} - \dfrac{44}{3}$

$\dfrac{x}{3} + \dfrac{y}{3} = 1$

$y = 3 - x$

Substituting this into the equation for the curve gives

$x + x(3-x) + 2(3-x)^2 = 6$

$x + 3x - x^2 + 18 - 12x + 2x^2 = 6$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

Therefore $x = 6$ is the only other point on the curve with the desired property. The coordinates at this point are $(6, -3)$.

1973 AB4/BC2

(a) What is the set of all values of b for which the graphs of $y = 2x + b$ and $y^2 = 4x$ intersect in two distinct points?

(b) In the case $b = -4$, find the area of the region enclosed by $y = 2x - 4$ and $y^2 = 4x$.

(c) In the case $b = 0$, find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 2x$ and $y^2 = 4x$.

1973 AB4/BC2
Solution

(a) The intersection occurs when

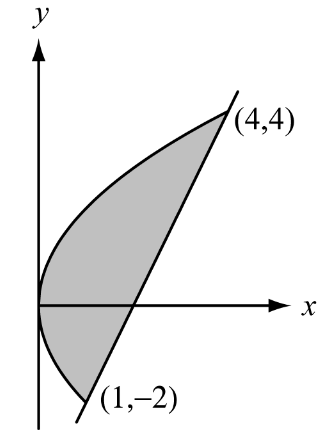
$(2x+b)^2 = 4x$

$4x^2 + 4bx + b^2 = 4x$

$4x^2 + 4(b-1)x + b^2 = 0$

The solutions to this equation are $x = \dfrac{-4(b-1) \pm \sqrt{16(b-1)^2 - 16b^2}}{8}$. There will be two distinct roots when $16(b-1)^2 - 16b^2 > 0$, this is, when $-32b + 16 > 0$. Hence there are two distinct roots when $b < \dfrac{1}{2}$.

(b) With $b = -4$, the intersection occurs when



$y = 2x - 4$

$y^2 = 4x$

$4x^2 - 16x + 16 = 4x$

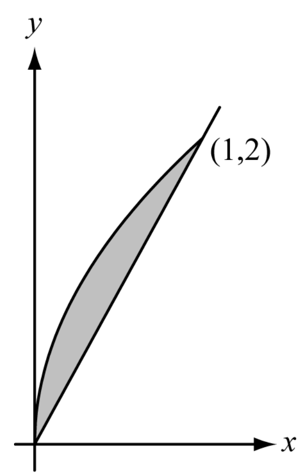
$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

Therefore the intersection points are $(1,-2)$ and $(4,4)$.

Area = $\displaystyle\int_{-2}^{4}\left(\frac{y+4}{2} - \frac{y^2}{4}\right)dy = \frac{y^2}{4} + 2y - \frac{y^3}{12}\Bigg|_{-2}^{4} = \left(4 + 8 - \frac{64}{12}\right) - \left(1 - 4 + \frac{8}{12}\right) = 9$

(c) When $b = 0$, intersection points are at $(0,0)$ and $(1,2)$.



Volume $= \pi\int_0^1 (4x - 4x^2)\,dx = \pi\left(2x^2 - \frac{4}{3}x^3\right)\Bigg|_0^1$

$= \pi\left(2 - \frac{4}{3}\right) = \frac{2\pi}{3}$

Or

Volume $= 2\pi\displaystyle\int_0^2 y\left(\frac{y}{2} - \frac{y^2}{4}\right)dy$

$= 2\pi\left(\frac{y^3}{6} - \frac{y^4}{16}\right)\Bigg|_0^2 = 2\pi\left(\frac{8}{6} - \frac{16}{16}\right) = \frac{2\pi}{3}$

1973 AB5/BC3

(a) Find the coordinates of the absolute maximum point for the curve $y = xe^{-kx}$ where k is a fixed positive number. Justify your answer.

(b) Write an equation for the set of absolute maximum points for the curves $y = xe^{-kx}$ as k varies through positive values.

1973 AB5/BC3
Solution

(a) $y' = -kxe^{-kx} + e^{-kx} = e^{-kx}(1-kx)$

Since $e^{-kx} \neq 0$, $y' = 0$ when $x = \dfrac{1}{k}$.

$y'' = k^2xe^{-kx} - ke^{-kx} - ke^{-kx} = e^{-kx}(k^2x - 2k)$

At $x = \dfrac{1}{k}$, $y'' = e^{-1}(k-2k) = e^{-1}(-k) < 0$ since k is positive. Therefore $x = \dfrac{1}{k}$ gives a relative maximum. Since $x = \dfrac{1}{k}$ is the only critical point, $x = \dfrac{1}{k}$ also gives the absolute maximum.

Therefore the coordinates of the absolute maximum point are $\left(\dfrac{1}{k}, \dfrac{1}{ke}\right)$.

(b) By part (a), $x = \dfrac{1}{k}$ and $y = \dfrac{1}{ke}$. Therefore $y = \dfrac{x}{e}$ for $x > 0$.

1973 AB6

A manufacturer finds it costs him $x^2 + 5x + 7$ dollars to produce x tons of an item. At production levels above 3 tons, he must hire additional workers, and his costs increase by $3(x-3)$ dollars on his total production. If the price he receives is \$13 per ton regardless of how much he manufactures and if he has a plant capacity of 10 tons, what level of output maximizes profits?

1973 AB6
Solution

For $0 \le x \le 3$, $P(x) = 13x - (x^2 + 5x + 7) = -x^2 + 8x - 7$

For $3 < x \le 10$, $P(x) = 13x - (x^2 + 5x + 7 + 3(x - 3)) = -x^2 + 5x + 2$

Profit function: $P(x) = \begin{cases} -x^2 + 8x - 7, & 0 \le x \le 3 \\ -x^2 + 5x + 2, & 3 < x \le 10 \end{cases}$

$$P'(x) = \begin{cases} -2x + 8, & 0 < x < 3 \\ -2x + 5, & 3 < x < 10 \end{cases}$$

$P'(x) \neq 0$ for any x in the interval $0 < x < 10$.
However $P'(x) > 0$ for $0 < x < 3$ and so $P(x)$ is increasing on this interval.
Also $P'(x) < 0$ for $3 < x < 10$ and so $P(x)$ is decreasing on this interval.
Therefore P has an absolute maximum at $x = 3$.

This can also be verified by checking the value at $x = 3$ against the values at the endpoints.

$P(0) = -7$
$P(10) = -48$
$P(3) = 8$

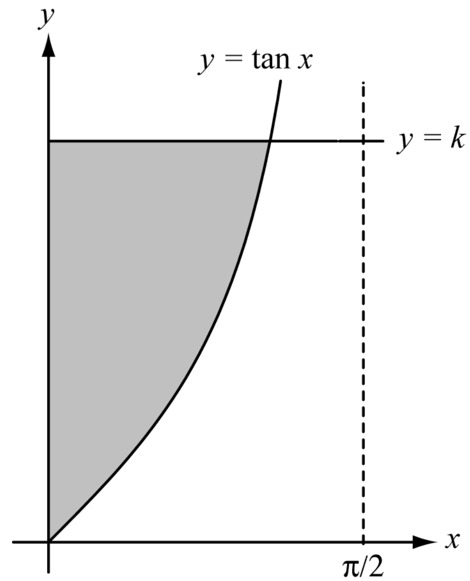
1973 AB7

(a) Find the area A, as a function of k, of the region in the first quadrant enclosed by the y-axis and the graphs of $y = \tan x$ and $y = k$ for $k > 0$.

(b) What is the value of A when $k = 1$?

(c) If the line $y = k$ is moving upward at the rate of $\frac{1}{10}$ units per second, at what rate is A changing when $k = 1$?

1973 AB7
Solution



(a) $A = \int_0^{\tan^{-1} k} (k - \tan x)\, dx = kx + \ln(\cos x)\Big|_0^{\tan^{-1} k}$

$= k \tan^{-1} k + \ln(\cos(\tan^{-1} k))$

$= k \tan^{-1} k + \ln \dfrac{1}{\sqrt{1+k^2}}$

or, using integration by parts,

$A = \int_0^k \tan^{-1} y\, dy = \left(y \tan^{-1} y - \dfrac{1}{2} \int \dfrac{2y}{1+y^2}\, dy \right)\Bigg|_0^k$

$= \left(y \tan^{-1} y - \dfrac{1}{2} \ln(1+y^2) \right)\Bigg|_0^k$

$= k \tan^{-1} k - \dfrac{1}{2} \ln(1+k^2)$

(b) When $k = 1$, $A = (1)\tan^{-1}(1) - \dfrac{1}{2}\ln(1+1) = \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$.

(c) By the chain rule, $\dfrac{dA}{dt} = \dfrac{dA}{dk} \cdot \dfrac{dk}{dt}$. We are given that $\dfrac{dk}{dt} = \dfrac{1}{10}$.

$\dfrac{dA}{dk} = k\left(\dfrac{1}{1+k^2}\right) + \tan^{-1} k - \dfrac{1}{2}\left(\dfrac{1}{1+k^2}\right)(2k)$

So at $k = 1$, $\dfrac{dA}{dk} = \dfrac{1}{2} + \dfrac{\pi}{4} - \dfrac{1}{2}\left(\dfrac{1}{2}\right)(2) = \dfrac{\pi}{4}$

Hence $\dfrac{dA}{dt} = \dfrac{\pi}{4}\left(\dfrac{1}{10}\right) = \dfrac{\pi}{40}$

1973 BC4

A kite flies according to the parametric equations
$x = \frac{t}{8}, \, y = -\frac{3}{64}t(t-128)$
where t is measured in seconds and $0 < t \le 90$.

(a) How high is the kite above the ground at time $t = 32$ seconds?

(b) At what rate is the kite rising at $t = 32$ seconds?

(c) At what rate is the string being reeled out at $t = 32$ seconds?

(d) At what time does the kite start to lose altitude?

1973 BC4
Solution

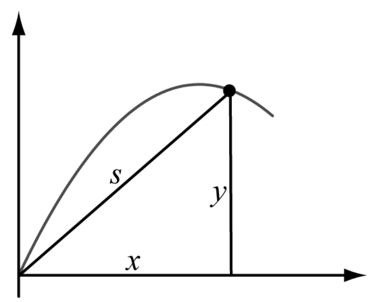
(a) $y = -\frac{3}{64}t(t-128)$

At $t = 32$, $y = -\frac{3}{64}(32)(32-128) = \left(-\frac{3}{2}\right)(-96) = 144$.

(b) $\frac{dy}{dt} = \left(-\frac{3}{64}t\right) + \left(-\frac{3}{64}\right)(t-128)$

At $t = 32$, $\frac{dy}{dt} = \left(-\frac{3}{2}\right) + \left(-\frac{3}{64}\right)(-96) = \left(-\frac{3}{2}\right) + \left(\frac{9}{2}\right) = 3$

(c) Let s be the length of the string.



$s = \sqrt{x^2 + y^2}$

$\frac{ds}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$

At $t = 32$, $x = 4$ and $y = 144$.

At $t = 32$, $\frac{dx}{dt} = \frac{1}{8}$ and $\frac{dy}{dt} = 3$.

Therefore at $t = 32$,

$$\frac{ds}{dt} = \frac{1}{2}\frac{1}{\sqrt{4^2 + 144^2}}\left(2(4)\frac{1}{8} + 2(144)(3)\right) = \frac{\frac{1}{2} + 432}{\sqrt{16 + (16 \cdot 9)^2}} = \frac{\frac{1}{2} + 432}{4\sqrt{1297}}.$$

(d) The kite will start to lose altitude at the instant when $0 = \frac{dy}{dt} = -\frac{6}{64}t + 6$. This occurs at $t = 64$ seconds.

1973 BC5

The distance x of a particle from a fixed point $x = 0$ is given by the differential equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 12x = 0$ for all time $t \geq 0$.

(a) Find the general solution of this differential equation.

(b) Find the solution of this differential equation satisfying the conditions $x = 1$ and $\frac{dx}{dt} = 2$ when $t = 0$.

(c) Does the particle pass through the point $x = 0$ after it begins moving? If so, determine the value of t at each such time; if not, explain.

1973 BC5
Solution

(a) The general solution can be found by computing the roots of the auxiliary equation.

$$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+12x=0$$

$$m^2+8m+12=0$$

$$(m+2)(m+6)=0$$

$$m=-2,\ m=-6$$

Therefore the general solution is $x=C_1e^{-2t}+C_2e^{-6t}$.

(b) When $t=0$, $x=1$ and $\frac{dx}{dt}=2$.

$$\frac{dx}{dt}=-2C_1e^{-2t}-6C_2e^{-6t}$$

The initial conditions give the two simultaneous equations:

$$\begin{cases}1=C_1+C_2\\ 2=-2C_1-6C_2\end{cases}$$

Hence $C_1=2$ and $C_2=-1$. Therefore $x=2e^{-2t}-e^{-6t}$.

(c) Can $x=0$?

$$0=2e^{-2t}-e^{-6t}$$

$$0=2e^{4t}-1$$

$$e^{4t}=\frac{1}{2}$$

$$4t=\ln\frac{1}{2}$$

$$t=\frac{1}{4}\ln\frac{1}{2}$$

But $\frac{1}{4}\ln\frac{1}{2}<0$ and hence the particle does not pass through the origin for any $t>0$.

1973 BC6

In each of the following cases, decide whether the infinite series converges. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k}$

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{\sin k}{k^{2}+\sqrt{k}}$

(c) $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k\ln^{2}k}$

1973 BC6
Solution

(a) Since $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e \neq 0$, the series must diverge. It is also sufficient to observe that for each term, $\left(1+\frac{1}{k}\right) > 1$ which would be enough to force the series to diverge.

(b) $\left|\frac{\sin k}{k^2+\sqrt{k}}\right| \leq \frac{1}{k^2+\sqrt{k}} < \frac{1}{k^2}$

The series converges absolutely by comparison with a convergent p-series ($p = 2$), and therefore the series converges.

(c) The terms of the series satisfy the conditions needed to use the integral test.

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{\ln x}\bigg|_2^b = \lim_{b\to\infty}\left(-\frac{1}{\ln b}+\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

Since the improper integral converges, so does the series.

1973 BC7

Given the definite integral $I = \int_0^{\pi} x\, f(\sin x)\, dx$, where f is a continuous function on the closed interval $[0,1]$.

(a) Use the substitution $x = \pi - y$ to show that $I = \frac{\pi}{2}\int_0^{\pi} f(\sin x) dx$.

(b) Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$.

1973 BC7
Solution

(a) Let $x = \pi - y$

$$I = \int_0^{\pi} x\, f(\sin x)\, dx$$
$$= \int_{\pi}^{0} (\pi - y)\, f(\sin(\pi - y))(-1)\, dy$$
$$= \int_0^{\pi} (\pi - y)\, f(\sin(\pi - y))\, dy$$
$$= \int_0^{\pi} (\pi - y)\, f(\sin y)\, dy$$
$$= \pi\int_0^{\pi} f(\sin y)\, dy - \int_0^{\pi} y\, f(\sin y)\, dy$$
$$= \pi\int_0^{\pi} f(\sin x)\, dx - I$$

Therefore $2I = \pi\int_0^{\pi} f(\sin x)dx$ and so $I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx$.

(b) Using part (a) with $f(x) = \frac{x}{2 - x^2}$, we have

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Method 1

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx = -\frac{\pi}{2}\arctan(\cos x)\Big|_0^{\pi} = -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{4}$$

Method 2

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx \quad (\text{let } u = \cos x)$$
$$= -\frac{\pi}{2}\int_1^{-1} \frac{1}{1+u^2}\, du = -\frac{\pi}{2}\arctan u\Big|_1^{-1} = -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{4}$$

1974 AB1/BC1

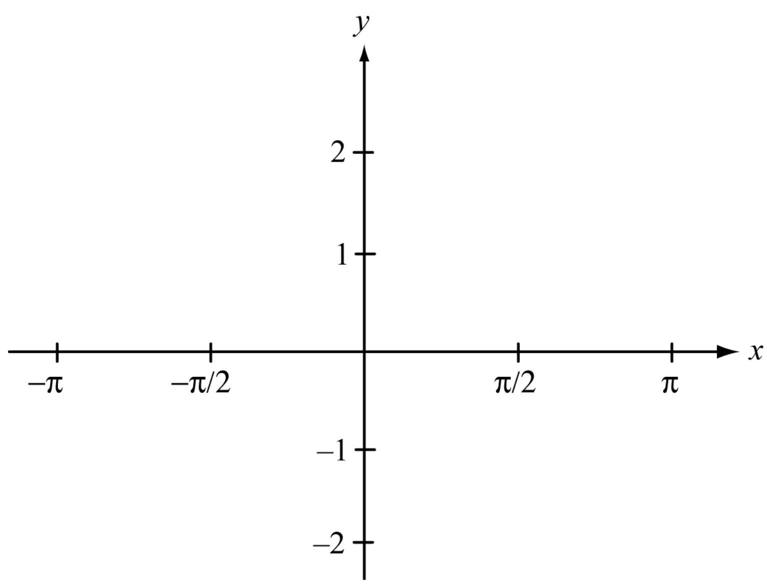
Given $f(x) = |\sin x|, -\pi \le x \le \pi$, and $g(x) = x^2$ for all real x.

(a) On the axes provided sketch the graph of *f*.

(b) Let $H(x) = g(f(x))$. Write an expression for $H(x)$.

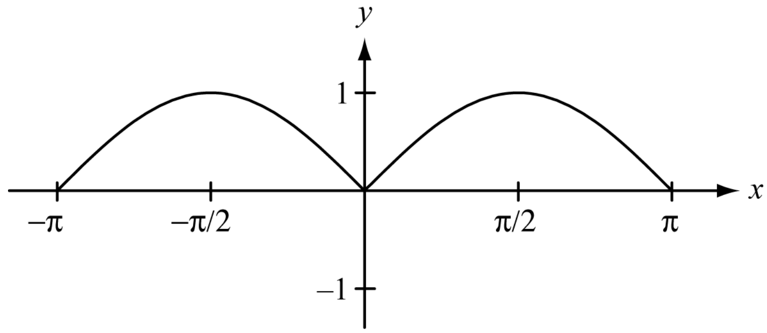
(c) Find the domain and range of *H*.

(d) Find and equation of the line tangent to the graph of H at the point where $x = \dfrac{\pi}{4}$.



1974 AB1/BC1
Solution

(a)



(b) $H(x) = \left(|\sin x|\right)^2 = \sin^2 x$

(c) The domain of H is $-\pi \le x \le \pi$

The range of H is $0 \le y \le 1$

(d) $H'(x) = 2\sin x \cos x$

$$H'\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1$$

$$H\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

Therefore the equation of the tangent line is

$$y - \frac{1}{2} = 1\left(x - \frac{\pi}{4}\right), \text{ or } y = x - \frac{\pi}{4} + \frac{1}{2}.$$

1974 AB2

Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$. The graph of $y = P(x)$ is symmetric with respect to the y-axis, has a relative maximum at $(0,1)$, and an absolute minimum at $(q,-3)$.

(a) Determine the values of $a, b, c,$ and d, and using these values write an expression for $P(x)$.

(b) Find all possible values of q.

1974 AB2
Solution

(a) $1 = P(0) = d$

$0 = P'(0) = c$

$P(-x) = P(x) \Rightarrow x^4 - ax^3 + bx^2 + 1 = x^4 + ax^3 + bx^2 + 1 \Rightarrow 2ax^3 = 0 \Rightarrow a = 0$

Alternatively, symmetry with respect to the y-axis means that only even powers of x will appear in the polynomial, and hence $a = c = 0$.

$P(x) = x^4 + bx^2 + 1$

$P(q) = q^4 + bq^2 + 1 = -3$

$P'(q) = 4q^3 + 2bq = 0$

Method 1:

$$\left.\begin{array}{l} 2q^4 + 2bq^2 + 8 = 0 \\ 4q^4 + 2bq^2 \qquad = 0 \end{array}\right\} \Rightarrow 2q^4 - 8 = 0 \Rightarrow q = \pm\sqrt{2}$$

Hence $16 + 4b = 0$ and so $b = -4$.

Method 2:

$2q(2q^2 + b) = 0 \Rightarrow 2q^2 = -b$ (note that q cannot equal 0.) Therefore $q^2 = -\dfrac{b}{2}$.

$\dfrac{b^2}{4} - \dfrac{b^2}{2} + 1 = -3 \Rightarrow -\dfrac{b^2}{4} = -4 \Rightarrow b^2 = 16 \Rightarrow b = -4$

(since $2q^2 = -b$ shows that $b < 0$)

Therefore $P(x) = x^4 - 4x^2 + 1$.

(b) $q^2 = -\dfrac{b}{2} = 2$

$q = \pm\sqrt{2}$

1974 AB3

Let $f(x) = kx^2 + c$.

(a) Find x_0 in terms of k such that the tangent lines to the graph of f at $(x_0, f(x_0))$ and $(-x_0, f(-x_0))$ are perpendicular.

(b) Find the slopes of the tangent lines mentioned in (a).

(c) Find the coordinates, in terms of k and c, of the point of intersection of the tangent lines mentioned in (a).

1974 AB3
Solution

(a) $f'(x) = 2kx$

The tangent lines will be perpendicular if
$-1 = f'(x_0)f'(-x_0) = (2kx_0)(-2kx_0) = -4k^2x_0^2$

$x_0^2 = \frac{1}{4k^2}$

$x_0 = \pm\frac{1}{2k}$

(b) $f'\left(\frac{1}{2k}\right) = 2k\left(\frac{1}{2k}\right) = 1$

$f'\left(-\frac{1}{2k}\right) = 2k\left(-\frac{1}{2k}\right) = -1$

(c) The equations of the two tangent lines are

$y - (kx_0^2 + c) = 1\cdot(x - x_0)$
$y - (kx_0^2 + c) = -1\cdot(x - (-x_0))$

The point of intersection is therefore at $x = 0$ since the left sides are equal.

At $x = 0$, $y = kx_0^2 + c - x_0 = k\left(\frac{1}{4k^2}\right) + c - \frac{1}{2k} = c - \frac{1}{4k}$.

The point of intersection is $\left(0, c - \frac{1}{4k}\right)$.

1974 AB4/BC2

Let f be a function defined for all $x > -5$ and having the following properties.

(i) $f''(x) = \dfrac{1}{3\sqrt{x+5}}$ for all x in the domain of f.

(ii) The line tangent to the graph of f at $(4,2)$ has an angle of inclination of $45°$.

Find an expression for $f(x)$.

1974 AB4/BC2
Solution

$$f'(x) = \int \frac{1}{3\sqrt{x+5}}\,dx = \frac{1}{3}\int (x+5)^{-1/2}\,dx = \frac{2}{3}(x+5)^{1/2} + C_1$$

$$1 = \tan 45° = f'(4) = \frac{2}{3}(9)^{1/2} + C_1 = 2 + C_1 \Rightarrow C_1 = -1$$

Therefore $f'(x) = \frac{2}{3}(x+5)^{1/2} - 1$.

$$f(x) = \int \left(\frac{2}{3}(x+5)^{1/2} - 1\right)dx = \frac{4}{9}(x+5)^{3/2} - x + C_2$$

$$2 = f(4) = \frac{4}{9}(9)^{3/2} - 4 + C_2 \Rightarrow C_2 = -6$$

Therefore $f(x) = \frac{4}{9}(x+5)^{3/2} - x - 6$.

1974 AB5/BC4

A ball is thrown from the origin of a coordinate system. The equation of its path is $y = mx - \frac{e^{2m}}{1000}x^2$, where m is positive and represents the slope of the path of the ball at the origin.

(a) For what value of m will the ball strike the horizontal axis at the greatest distance from the origin? Justify your answer.

(b) For what value of m will the ball strike at the greatest height on a vertical wall located 100 feet from the origin?

1974 AB5/BC4
Solution

(a) The ball will strike the horizontal axis when $y = 0$.

$$mx - \frac{e^{2m}}{1{,}000}x^2 = 0 \Rightarrow x = 1000me^{-2m}$$

$$\frac{dx}{dm} = 1000e^{-2m} - 2000me^{-2m} = 1000e^{-2m}(1-2m) = 0 \text{ when } m = \frac{1}{2}.$$

$\frac{dx}{dm} > 0$ for $m < \frac{1}{2}$ and $\frac{dx}{dm} < 0$ for $m > \frac{1}{2}$. Therefore $m = \frac{1}{2}$ gives the maximum distance from the origin.

Alternatively, $\frac{d^2y}{dm^2} = 1000(-2e^{-2m} + (1-2m)(-2)e^{-2m}) = -4000e^{-2m}(1-m)$. At $m = \frac{1}{2}$, $\frac{d^2x}{dm^2} = -2000e^{-1} < 0$. Therefore $m = \frac{1}{2}$ gives a relative maximum. But since this is the only critical point, it is also an absolute maximum.

(b) When $x = 100$, $y = 100m - \frac{e^{2m}}{1000}(100)^2 = 100m - 10e^{2m}$. We want the value of m that makes y a maximum.

$$\frac{dy}{dm} = 100 - 20e^{2m} = 0$$

$$5 = e^{2m}$$

$$m = \frac{\ln 5}{2}$$

1974 AB6

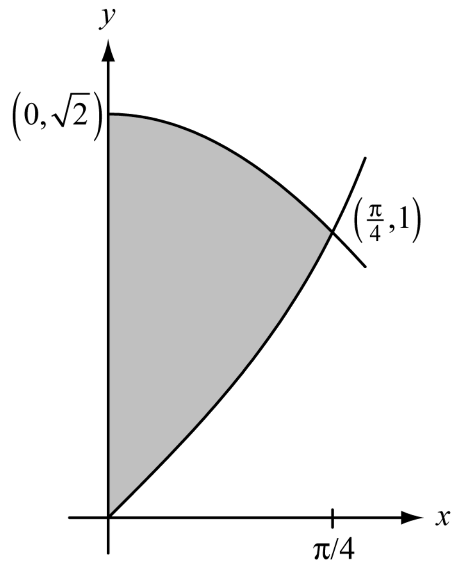
Given two functions f and g defined by $f(x) = \tan x$ and $g(x) = \sqrt{2}\cos x$.

(a) Find the coordinates of the point of intersection of the graphs of f and g in the interval $0 < x < \frac{\pi}{2}$.

(b) Find the area of the region enclosed by the <u>y-axis</u> and the graphs of f and g.

1974 AB6
Solution

(a) At the intersection



$\tan x = \sqrt{2}\cos x$

$\sin x = \sqrt{2}\cos^2 x = \sqrt{2}(1 - \sin^2 x)$

$\sqrt{2}\sin^2 x + \sin x - \sqrt{2} = 0$

$(\sqrt{2}\sin x - 1)(\sin x + \sqrt{2}) = 0$

The only possibility is $\sin x = \dfrac{1}{\sqrt{2}}$. The solution in the interval $0 < x < \dfrac{\pi}{2}$ is at $x = \dfrac{\pi}{4}$. So the point of intersection is $\left(\dfrac{\pi}{4}, 1\right)$.

(b) Area $= \int_0^{\pi/4} (\sqrt{2}\cos x - \tan x)\, dx$

$= (\sqrt{2}\sin x + \ln \cos x)\Big|_0^{\pi/4}$

$= 1 + \ln \dfrac{\sqrt{2}}{2} = 1 - \ln \sqrt{2}$

1974 AB7

The rate of change in the number of bacteria in a culture is proportional to the number present. In a certain laboratory experiment, a culture had 10,000 bacteria initially, 20,000 bacteria at time t_1 minutes, and 100,000 bacteria at $(t_1 + 10)$ minutes.

(a) In terms of t only, find the number of bacteria in the culture at any time t minutes, $t \geq 0$.

(b) How many bacteria were there after 20 minutes?

(c) How many minutes had elapsed when the 20,000 bacteria were observed?

1974 AB7
Solution

(a) Let N = number of bacteria present at time t. Then

$$\frac{dN}{dt} = kN$$
$$\frac{dN}{N} = kdt$$
$$\ln N = kt + C_1$$
$$N = C_2 e^{kt}$$

At $t = 0$, $10000 = C_2 e^{k\cdot 0} = C_2$. Therefore $N = 10000e^{kt}$.

At $t = t_1$, $20000 = 10000e^{kt_1} \Rightarrow 2 = e^{kt_1}$

At $t = t_1 + 10$, $100000 = 10000e^{k(t_1+10)} \Rightarrow 10 = e^{kt_1} \cdot e^{10k} = 2e^{10k}$

Therefore $k = \frac{\ln 5}{10}$ and so $N = 10000e^{\frac{\ln 5}{10}t}$.

(b) At $t = 20$, $N = 10000e^{\frac{\ln 5}{10}(20)} = 10000e^{\ln 25} = 250000$

(c) $20000 = 10000e^{\frac{\ln 5}{10}t}$

$$2 = e^{\frac{\ln 5}{10}t}$$

$$\ln 2 = \frac{\ln 5}{10}t$$

$$t = \frac{10\ln 2}{\ln 5}$$

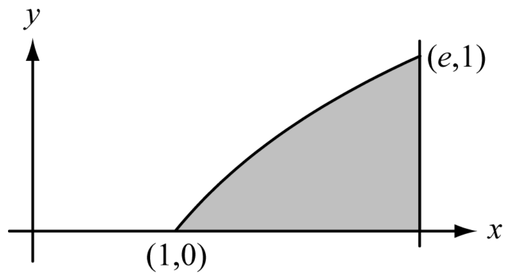
1974 BC3

Let R be the region bounded by the graph of $y = \ln x$, the line $x = e$, and the x-axis.

(a) Find the volume generated by revolving R about the <u>x-axis</u>.

(b) Find the volume generated by revolving R about the <u>y-axis</u>.

1974 BC3
Solution



(a) Shells:

$$\text{Volume} = 2\pi\int_0^1 (e-e^y)\,y\,dy$$

$$= 2\pi\int_0^1 (ey - ye^y)\,dy \quad \text{(integrate by parts)}$$

$$= 2\pi\left(\frac{ey^2}{2} - (ye^y - e^y)\right)\Bigg|_0^1 = 2\pi\left(\frac{e}{2}-1\right) = \pi(e-2)$$

Disks:

$$\text{Volume} = \pi\int_1^e (\ln x)^2\,dx \quad \text{(integrate by parts twice)}$$

$$= \pi\left(x(\ln x)^2 - 2x\ln x + 2x\right)\Big|_1^e = \pi(e-2e+2e-2) = \pi(e-2)$$

(b) Disks:

$$\text{Volume} = \pi\int_0^1 (e^2 - e^{2y})\,dy = \pi\left(e^2 y - \frac{e^{2y}}{2}\right)\Bigg|_0^1 = \pi\left(\frac{e^2}{2}+\frac{1}{2}\right) = \frac{\pi}{2}(e^2+1)$$

Shells:

$$\text{Volume} = 2\pi\int_1^e x\ln x\,dx \quad \text{(integrate by parts)}$$

$$= 2\pi\left(\frac{x^2\ln x}{2} - \frac{x^2}{4}\right)\Bigg|_1^e = 2\pi\left(\frac{e^2}{2}-\frac{e^2}{4}+\frac{1}{4}\right) = 2\pi\left(\frac{e^2}{4}+\frac{1}{4}\right) = \frac{\pi}{2}(e^2+1)$$

1974 BC5

Given the parametric equations $x = 2(\theta - \sin\theta)$ and $y = 2(1 - \cos\theta)$:

(a) Find $\frac{dy}{dx}$ in terms of θ.

(b) Find an equation of the line tangent to the graph at $\theta = \pi$.

(c) Find an equation of the line tangent to the graph at $\theta = 2\pi$.

(d) Set up but do not evaluate an integral representing the length of the curve over the interval $0 \le \theta \le 2\pi$. Express the integrand as a function of θ.

1974 BC5
Solution

(a) $\dfrac{dx}{d\theta} = 2 - 2\cos\theta,\quad \dfrac{dy}{d\theta} = 2\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sin\theta}{2-2\cos\theta} = \frac{\sin\theta}{1-\cos\theta}$$

(b) At $\theta = \pi,\ \dfrac{dy}{dx} = \dfrac{\sin\pi}{1-\cos\pi} = 0$. This means the tangent line is horizontal.

At $\theta = \pi,\ y = 4$, so the equation of the tangent line is $y = 4$.

(c) At $\theta = 2\pi,\ \dfrac{dy}{dx} = \dfrac{\sin 2\pi}{1-\cos 2\pi}$ which is not defined.

$$\lim_{\theta\to 2\pi} \frac{\sin\theta}{1-\cos\theta} = \lim_{\theta\to 2\pi} \frac{\cos\theta}{\sin\theta} \to \infty$$

This means that there is a vertical tangent. At $\theta = 2\pi$, $x = 4\pi$, so the equation of the tangent line is $x = 4\pi$

(d) Length $= \int_0^{2\pi} \sqrt{(2-2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta = 2\int_0^{2\pi} \sqrt{2-2\cos\theta}\, d\theta$

1974 BC6

Given the differential equation $y'' + 4y' + 5y = 0$.

(a) Find the general solution $y(x)$.

(b) Find the solution that satisfies the conditions $y(0) = 0$ and $y'(0) = 1$.

(c) Evaluate the limit of the solution determined in (b) as x increases without bound. Justify your answer.

1974 BC6
Solution

(a) $y'' + 4y' + 5y = 0$

The auxiliary equation is $m^2 + 4m + 5 = 0$.

$$m = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

So $y = C_1 e^{-2x} \cos x + C_2 e^{-2x} \sin x$

(b) $0 = y(0) = C_1$

$y = C_2 e^{-2x} \sin x$
$y' = C_2 (e^{-2x} \cos x - 2e^{-2x} \sin x)$

$1 = y'(0) = C_2$

So $y = e^{-2x} \sin x$

(c) $\lim_{x \to \infty} e^{-2x} \sin x = 0$ since $\lim_{x \to \infty} e^{-2x} = 0$ and $\sin x$ is bounded between -1 and 1.

1974 BC7

(a) Consider a function f such that $f'(x)$ exists for all real x. Suppose that:

(i) $f'(r) = 0$ and $f'(s) = 0$
(ii) $f'(x) \neq 0$ for all x in the interval $r < x < s$; that is, r and s are consecutive roots of $f'(x) = 0$.

Prove that there is at most one root of $f(x) = 0$ in $r < x < s$.

(b) Given a function f such that $f'(x) = (4 - x^2)(4 + x^2)e^{-x^2}$. What is the maximum number of real roots of $f(x) = 0$ in $-2 < x < 2$? Justify your answer.

1974 BC7
Solution

(a) Suppose there are x_1, x_2 with $f(x_1) = f(x_2) = 0,\ r < x_1 < x_2 < s$. Since f is differentiable on (x_1, x_2), one may use Rolle's Theorem or the Mean Value Theorem to conclude that there exists x^* with $x_1 < x^* < x_2$ such that $f'(x^*) = 0$. This contradicts the fact that $f' \neq 0$ on (r, s). Hence there is at most one root of f in (r, s).

(b) The function f is differentiable for all real x. Since $f'(-2) = f'(2) = 0$, we can take $r = -2$ and $s = 2$ in part (a). Since –2 and 2 are the only roots of f', they are consecutive roots. Therefore condition (ii) holds and hence the conclusion of part (a) applies. Thus, the maximum number of real roots of $f(x) = 0$ in $-2 < x < 2$ is one.

1975 AB1

Given the function f defined by $f(x) = \ln(x^2 - 9)$.

(a) Describe the symmetry of the graph of f.

(b) Find the domain of f.

(c) Find all values of x such that $f(x) = 0$.

(d) Write a formula for $f^{-1}(x)$, the inverse function of f, for $x > 3$.

1975 AB1
Solution

(a) $f(-x) = \ln((-x)^2 - 9) = \ln(x^2 - 9) = f(x)$
Therefore the graph of f is symmetric with respect to the y-axis.

(b) Since we need $x^2 - 9 > 0$, the domain of f is the set $\{x \mid x < -3 \text{ or } x > 3\}$

(c) $f(x) = 0$ when $x^2 - 9 = 1$. This happens for $x = \pm\sqrt{10}$.

(d) Method 1:

$f(x) = \ln(x^2 - 9) \Rightarrow x^2 - 9 = e^{f(x)} = e^y$
Since $x > 3$, $x = \sqrt{e^y + 9}$.

Hence $f^{-1}(x) = \sqrt{e^x + 9}$.

Method 2:

$y = \ln(x^2 - 9)$, so interchanging variables gives $x = \ln(y^2 - 9)$.
$e^x = y^2 - 9$
$y = \sqrt{e^x + 9}$

Hence $f^{-1}(x) = \sqrt{e^x + 9}$.

1975 AB2

A particle moves along the x-axis in such a way that its position at time t for $t \geq 0$ is given by $x = \frac{1}{3}t^3 - 3t^2 + 8t$.

(a) Show that at time $t = 0$, the particle is moving to the right.

(b) Find all values of t for which the particle is moving to the left.

(c) What is the position of the particle at time $t = 3$?

(d) When $t = 3$, what is the total distance the particle has traveled?

1975 AB2
Solution

(a) $v = \dfrac{dx}{dt} = t^2 - 6t + 8$

$v(0) = 8 > 0$ and so the particle is moving to the right at $t = 0$.

(b) The particle is moving to the left when $v(t) = t^2 - 6t + 8 = (t-4)(t-2) < 0$.
Therefore the particle moves to the left for $2 < t < 4$.

(c) At time $t = 3$, $x = \dfrac{1}{3}(3)^3 - 3(3)^2 + 8(3) = 6$.

(d) The particle changes direction at $t = 2$.

$x(0) = 0$

$x(2) = \dfrac{1}{3}(2)^3 - 3(2)^2 + 8(2) = \dfrac{20}{3}$

$x(3) = 6$

Distance = $(x(2) - x(0)) + (x(2) - x(3)) = \dfrac{20}{3} + \dfrac{2}{3} = \dfrac{22}{3}$

1975 AB3

Given the function f defined for all real numbers x by $f(x) = 2|x-1|x^2$.

(a) What is the range of the function?

(b) For what values of x is the function continuous?

(c) For what values of x is the derivative of $f(x)$ continuous?

(d) Determine the value of $I = \int_0^1 f(x)\,dx$.

1975 AB3
Solution

(a) The range of f is the set $\{y \mid y \geq 0\}$.

(b) The function is continuous for all real numbers.

(c) For $x > 1$, $f(x) = 2(x-1)x^2$ and hence $f'(x) = 6x^2 - 4x$.

For $x < 1$, $f(x) = 2(1-x)x^2$ and hence $f'(x) = -6x^2 + 4x$.

So $\lim_{x \to 1^+} f'(x) = 2$ but $\lim_{x \to 1^-} f'(x) = -2$. Therefore f' is not continuous at $x = 1$.

The derivative is continuous at all other values of x, however.

(d) For $0 \leq x \leq 1$, $f(x) = 2(1-x)x^2$. Therefore

$$\int_0^1 f(x)\,dx = \int_0^1 2(1-x)x^2\,dx = \int_0^1 (2x^2 - 2x^3)\,dx = \left(\frac{2x^3}{3} - \frac{x^4}{2}\right)\Bigg|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

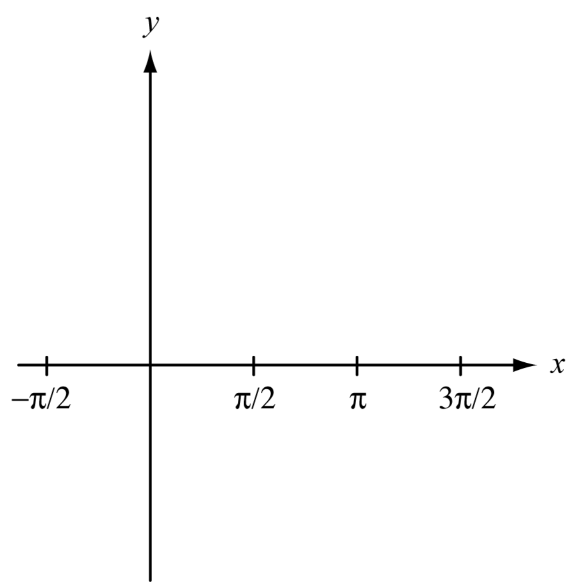
1975 AB4/BC1

Given the function defined by $y = x + \sin x$ for all x such that $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

(a) Find the coordinates of <u>all</u> maximum and minimum points on the given interval. Justify your answers.

(b) Find the coordinates of <u>all</u> points of inflection on the given interval. Justify your answers.

(c) On the axes provided, sketch the graph of the function.



1975 AB4/BC1
Solution

(a) $y' = 1 + \cos x$

Therefore $x = \pi$ is the only critical point on the interval $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. But $y' \ge 0$ on this interval, hence π is not an extreme point. The minimum and maximum must occur at the endpoints.

At $x = -\frac{\pi}{2}$, $y = -\frac{\pi}{2} + \sin\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} - 1$

At $x = \frac{3\pi}{2}$, $y = \frac{3\pi}{2} + \sin\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} - 1$

The absolute minimum is at $\left(-\frac{\pi}{2}, -\frac{\pi}{2} - 1\right)$.

The absolute maximum is at $\left(\frac{3\pi}{2}, \frac{3\pi}{2} - 1\right)$.

(b) $y'' = -\sin x$

$y'' = 0$ at $x = 0$ and $x = \pi$.

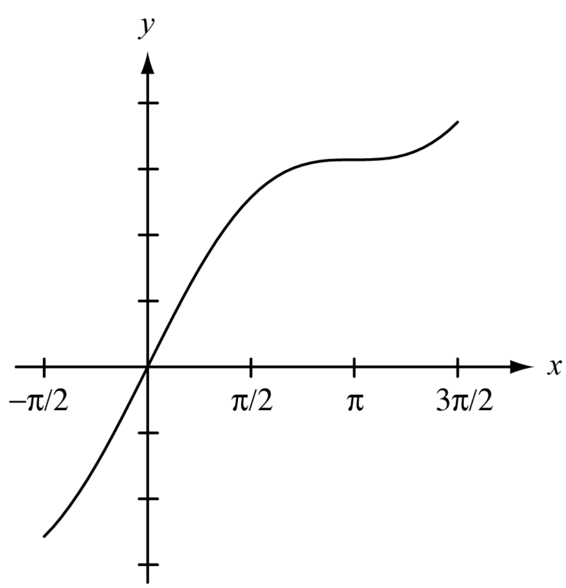
$y'' > 0$ for $-\frac{\pi}{2} < x < 0$

$y'' < 0$ for $0 < x < \pi$

$y'' > 0$ for $\pi < x < \frac{3\pi}{2}$

Therefore $(0,0)$ and (π,π) are inflection points.

(c)



1975 AB5

The line $x = c$ where $c > 0$ intersects the cubic $y = 2x^3 + 3x^2 - 9$ at point P and the parabola $y = 4x^2 + 4x + 5$ at point Q.

(a) If a line tangent to the cubic at point P is parallel to the line tangent to the parabola at point Q, find the value of c where $c > 0$.

(b) Write the equations of the two tangent lines described in (a).

1975 AB5
Solution

(a) $y' = 6x^2 + 6x$ for the cubic

The slope of the tangent line at point P, where $x = c$, is $6c^2 + 6c$.

$y' = 8x + 4$ for the parabola

The slope of the tangent line at point Q, where $x = c$, is $8c + 4$.

Since the two lines are parallel,

$6c^2 + 6c = 8c + 4$

$6c^2 - 2c - 4 = 0$

$2(3c + 2)(c - 1) = 0$

Since $c > 0$, the solution is $c = 1$.

(b) tangent to cubic:

The slope is $m = 12$ and the line contains $(1, -4)$.
Therefore the equation is $y + 4 = 12(x - 1)$, or $y = 12x - 16$.

tangent to parabola:

The slope is $m = 12$ and the line contains $(1, 13)$.
Therefore the equation is $y - 13 = 12(x - 1)$, or $y = 12x + 1$.

1975 AB6/BC2

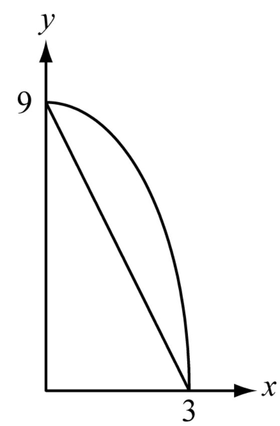
Let R be the region in the first quadrant bounded by the graphs of $\frac{x^2}{9} + \frac{y^2}{81} = 1$ and $3x + y = 9$.

(a) Set up but do not evaluate an integral representing the area of R. Express the integrand as a function of a single variable.

(b) Set up but do not evaluate an integral representing the volume of the solid generated when R is rotated about the x-axis. Express the integrand as a function of a single variable.

(c) Set up but do not evaluate an integral representing the volume of the solid generated when R is rotated about the y-axis. Express the integrand as a function of a single variable.

1975 AB6/BC2
Solution



The graphs can be written as

$y=\sqrt{81-9x^2}$ and $y=9-3x$

or

$x=\sqrt{9-\dfrac{y^2}{9}}$ and $x=3-\dfrac{y}{3}$.

(a) Area $=\int_0^3 (\sqrt{81-9x^2}-(9-3x))\,dx$

or

Area $=\displaystyle\int_0^9 \sqrt{9-\frac{y^2}{9}}-\left(3-\frac{y}{3}\right)dy$

(b) <u>Disks</u>:

Volume = $\pi\int_0^3 ((\sqrt{81-9x^2})^2-(9-3x)^2)\,dx$

or

<u>Shells</u>:

Volume = $2\pi\displaystyle\int_0^9 y\left(\sqrt{9-\frac{y^2}{9}}-\left(3-\frac{y}{3}\right)\right)dy$

(c) <u>Disks</u>:

Volume = $\pi\displaystyle\int_0^9 \left(\left(\sqrt{9-\frac{y^2}{9}}\right)^2-\left(3-\frac{y}{3}\right)^2\right)dy$

or

<u>Shells</u>:

Volume = $2\pi\int_0^3 x(\sqrt{81-9x^2}-(9-3x))\,dx$

1975 AB7/BC5

Given a function f with the following properties:

(i) $f(x+h) = e^h f(x) + e^x f(h)$ for all real numbers x and h

(ii) $f(x)$ has a derivative for all real numbers x

(iii) $f'(0) = 2$

(a) Show that $f(0) = 0$.

(b) Using the definition of the $f'(0)$, find $\lim_{x\to 0} \frac{f(x)}{x}$.

(c) Prove there exists a real number p such that $f'(x) = f(x) + pe^x$ for all real numbers x.

(d) What is the value of the number p described in (c)?

1975 AB7/BC5
Solution

(a) Let $x = 0$ and $h = 0$ in (i). Then $f(0) = e^0 f(0) + e^0 f(0) = 2f(0)$. Therefore $f(0) = 0$.

(b) $2 = f'(0) = \lim_{x\to 0} \frac{f(0+x) - f(0)}{x} = \lim_{x\to 0} \frac{e^0 f(x) + e^x f(0) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^h f(x) + e^x f(h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(e^h - 1) f(x)}{h} + \lim_{h\to 0} \frac{e^x f(h)}{h}$$

$$= f(x)\left(\lim_{h\to 0} \frac{e^h - 1}{h}\right) + \left(\lim_{h\to 0} \frac{f(h)}{h}\right) e^x$$

$$= f(x) + pe^x, \text{ where } p = \lim_{h\to 0} \frac{f(h)}{h}.$$

(d) $p = 2$ from part (b).

This can also be deduced from part (a) and property (iii) even if part (b) or part (c) could not be done. Substituting $x = 0$ into the relationship described in part (c) gives

$$2 = f'(0) = f(0) + pe^0 = 0 + p = p$$

1975 BC3

A particle moves on the circle $x^2 + y^2 = 1$ so that at time $t \geq 0$ the position is given by the vector $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$.

(a) Find the velocity vector.

(b) Is the particle ever at rest? Justify your answer.

(c) Give the coordinates of the point that the particle approaches as t increases without bound.

1975 BC3
Solution

(a) $\dfrac{dx}{dt} = \dfrac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2} = \dfrac{-4t}{(1+t^2)^2}$

$\dfrac{dy}{dt} = \dfrac{(1+t^2)(2)-2t(2t)}{(1+t^2)^2} = \dfrac{2-2t^2}{(1+t^2)^2}$

The velocity vector is $\left(\dfrac{-4t}{(1+t^2)^2}, \dfrac{2-2t^2}{(1+t^2)^2}\right)$

(b) $\dfrac{dx}{dt} = 0$ at $t = 0$ and $\dfrac{dy}{dt} = 0$ at $t = 1$. Therefore there is no t such that $\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0$ at the same time. Hence the particle is never at rest.

(c) $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} \dfrac{1-t^2}{1+t^2} = \lim\limits_{t\to\infty} \dfrac{\frac{1}{t^2}-1}{\frac{1}{t^2}+1} = -1$

$\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} \dfrac{2t}{1+t^2} = \lim\limits_{t\to\infty} \dfrac{2}{\frac{1}{t}+t} = 0$

Hence the particle approaches the point $(-1, 0)$ as t increases without bound.

1975 BC4

(a) Determine whether the series $\frac{1}{3} - \frac{2^3}{3^2} + \frac{3^3}{3^3} - \frac{4^3}{3^4} + \cdots + \frac{(-1)^{n-1} n^3}{3^n} + \cdots$ is convergent. Justify your answer.

(b) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$. Justify your answer.

1975 BC4
Solution

(a) Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-1)^n \dfrac{(n+1)^3}{3^{n+1}}}{(-1)^{n-1}\dfrac{n^3}{3^n}}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3} < 1$$

Therefore the series converges absolutely, and hence converges.

or

Alternating series test:

(i) the terms alternate

(ii) $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\dfrac{n^3}{3^n} = 0$

(iii) $\dfrac{(n+1)^3}{3^{n+1}} < \dfrac{n^3}{3^n}$ for $n > 2$ since $(n+1)^3 < 3n^3$ for $n > 2$.

Therefore $|a_{n+1}| < |a_n|$ for $n > 2$.

Since all three conditions hold, the series converges.

(b) $$\lim_{n\to\infty}\left|\frac{\dfrac{(x-2)^{n+1}}{(n+1)3^{n+1}}}{\dfrac{(x-2)^n}{n3^n}}\right| = \lim_{n\to\infty}\left|\frac{x-2}{3}\cdot\frac{n}{n+1}\right| = \frac{|x-2|}{3}$$

The series converges if $\dfrac{|x-2|}{3} < 1$, hence for $-1 < x < 5$.

At $x = 5$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ which diverges (harmonic series).

At $x = -1$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges (alternating harmonic series).

Therefore the interval of convergence is $-1 \le x < 5$.

1975 BC6

Given the differential equation $y'' - (p+1)y' + py = 0$, where p is a real number not equal to 1.

(a) Find the general solution of the differential equation.

(b) Find the particular solution satisfying the conditions $y(0) = 0$ and $y'(0) = 1$.

(c) Find the limit of the solution in (b) as $p \to 1$. Justify your answer.

1975 BC6
Solution

(a) $r^2 - (p+1)r + p = 0$

$(r-p)(r-1) = 0$

$r = p,\ r = 1$

The general solution is $y = C_1 e^{px} + C_2 e^x$.

(b) $y' = pC_1 e^{px} + C_2 e^x$

$y(0) = 0 = C_1 + C_2$

$y'(0) = 1 = pC_1 + C_2$

$C_2 = -C_1$

$1 = (p-1)C_1$

$C_1 = \dfrac{1}{p-1}, C_2 = -\dfrac{1}{p-1}$

The particular solution is $y = \dfrac{1}{p-1}e^{px} - \dfrac{1}{p-1}e^x = \dfrac{e^{px}-e^x}{p-1}$.

(c) $$\lim_{p\to 1}\frac{e^{px}-e^x}{p-1} = \lim_{p\to 1}\frac{\frac{d}{dp}(e^{px}-e^x)}{\frac{d}{dp}(p-1)} = \lim_{p\to 1}\frac{xe^{px}}{1} = xe^x$$

(using L'Hôpital's rule with respect to p as the variable.)

1975 BC7

(a) For what value of m is the line $y = mx$ tangent to the graph of $y = \ln x$?

(b) Prove that the graph of $y = \ln x$ lies entirely below the graph of the line found in (a).

(c) Use the results of (b) to show that $e^x \geq x^e$ for $x > 0$.

1975 BC7
Solution

(a) The line and the graph must intersect, so $mx = \ln x$.

The slopes must be the same, so $m = \frac{1}{x}$.

So $1 = \ln x$ which means that $x = e$ and $m = \frac{1}{e}$.

(b) Let $h(x) = \frac{x}{e} - \ln x$ for $x > 0$. Then $h'(x) = \frac{1}{e} - \frac{1}{x}$.

$h'(x) = 0$ for $x = e$. Since $h''(x) = \frac{1}{x^2} > 0$ for all $x > 0$, there is an absolute minimum at $x = e$. But $h(e) = 0$ and therefore $\frac{x}{e} - \ln x \geq 0$ for all $x > 0$. This shows that $\frac{x}{e} \geq \ln x$ for all $x > 0$.

(c) Since $\frac{x}{e} \geq \ln x$ for all $x > 0$, it is also true that $x \geq e \ln x$ for all $x > 0$.

Since the exponential function is an increasing function, this last inequality implies that $e^x \geq e^{e \ln x} = e^{\ln x^e} = x^e$ for all $x > 0$.

1976 AB1

Let f be the real-valued function defined by $f(x) = \sqrt{1+6x}$.

(a) Give the domain and range of f.

(b) Determine the slope of the line tangent to the graph of f at $x = 4$.

(c) Determine the y-intercept of the line tangent to the graph of f at $x = 4$.

(d) Give the coordinates of the point on the graph of f where the tangent line is parallel to $y = x + 12$.

1976 AB1
Solution

(a) The domain of f is $x \geq -\frac{1}{6}$.

The range of f is $y \geq 0$.

(b) $f'(x) = \dfrac{3}{\sqrt{1+6x}}$

The slope of the tangent line at $x = 4$ is $f'(4) = \frac{3}{5}$.

(c) $f(4) = 5$

The tangent line is $y - 5 = \frac{3}{5}(x-4)$

Therefore the y-intercept is at $y = \frac{13}{5}$.

(d) The tangent line parallel to $y = x + 12$ has slope 1.

$$f'(x) = \frac{3}{\sqrt{1+6x}} = 1$$

$$9 = 1 + 6x$$

$$x = \frac{4}{3}$$

$$y = \sqrt{1 + 6\left(\frac{4}{3}\right)} = 3$$

The coordinates of the point are $\left(\frac{4}{3}, 3\right)$.

1976 AB2

Given the two functions f and h such that $f(x) = x^3 - 3x^2 - 4x + 12$ and

$$h(x) = \begin{cases} \dfrac{f(x)}{x-3} & \text{for } x \neq 3 \\ p & \text{for } x = 3. \end{cases}$$

(a) Find all zeros of the function f.

(b) Find the value of p so that the function h is continuous at $x = 3$. Justify your answer.

(c) Using the value of p found in part (b), determine whether h is an even function. Justify your answer.

1976 AB2
Solution

(a) The zeros of $f(x) = x^3 - 3x^2 - 4x + 12$ can be found by factoring. The rational candidates are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. Since $f(2) = 0$, long (or synthetic) division gives

2	1	–3	–4	12
		2	–2	–12
	1	–1	–6	0

Therefore $f(x) = (x-2)(x^2 - x - 6) = (x-2)(x+2)(x-3)$ and so the zeros are $x = -2, 2, 3$.

(b) For continuity we need that $h(3) = \lim_{x\to 3} h(x)$.

$$\lim_{x\to 3}\frac{f(x)}{x-3} = \lim_{x\to 3}\frac{(x-2)(x+2)(x-3)}{x-3} = \lim_{x\to 3}(x-2)(x+2) = 5.$$

Therefore $p = 5$ will make the function h continuous at $x = 3$.

or

$h(x) = x^2 - 4$ for $x \neq 3$. When $x = 3$, $x^2 - 4 = 5$. So if $p = 5$, then $h(x) = x^2 - 4$ for all x and is therefore continuous.

(c) $h(x) = \begin{cases} x^2 - 4 & x \neq 3 \\ 5 & x = 3 \end{cases}$

$$h(-x) = \begin{cases} (-x)^2 - 4 = x^2 - 4 & x \neq 3 \\ 5 & x = 3 \end{cases}$$

$$= h(x)$$

Therefore h is an even function.

or

With $p = 5$, $h(x) = x^2 - 4$ for all x and h is therefore an even function.

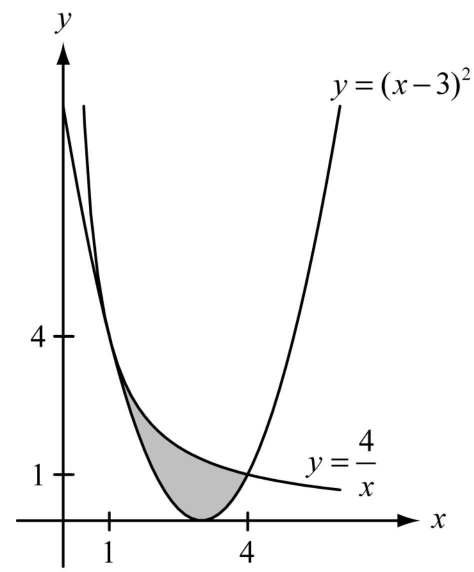
1976 AB3/BC2

Let R be the region bounded by the curves $f(x)=\frac{4}{x}$ and $g(x)=(x-3)^2$.

(a) Find the area of R.

(b) Find the volume of the solid generated by revolving R about the <u>x-axis</u>.

1976 AB3/BC2
Solution



Intersection points occur when

$\frac{4}{x}=(x-3)^2$

$0=x^3-6x^2+9x-4=(x-4)(x-1)^2$

Thus the intersection points are at $(1,4)$ and $(4,1)$.

(a) Area $=\int_1^4\left(\frac{4}{x}-(x-3)^2\right)dx$

$$=\left(4\ln x-\frac{(x-3)^3}{3}\right)\Bigg|_1^4=4\ln 4-3$$

(b) Volume $=\int_1^4\pi\left(\left(\frac{4}{x}\right)^2-(x-3)^4\right)dx$

$$=\pi\left(-\frac{16}{x}-\frac{(x-3)^5}{5}\right)\Bigg|_1^4=\pi\left(\left(-4-\frac{1}{5}\right)-\left(-16-\frac{-32}{5}\right)\right)$$

$$=\frac{27\pi}{5}$$

or

Volume $=2\pi\int_1^4 y\left(\frac{4}{y}-\left(3-\sqrt{y}\right)\right)dy+2\pi\int_0^1 y\left(\left(3+\sqrt{y}\right)-\left(3-\sqrt{y}\right)\right)dy$

$$=2\pi\left(4y-\frac{3y^2}{2}+\frac{2y^{5/2}}{5}\right)\Bigg|_1^4+2\pi\left(2\cdot\frac{2y^{5/2}}{5}\right)\Bigg|_0^1$$

$$=\pi\left(-21+\frac{124}{5}\right)+\pi\left(\frac{8}{5}\right)$$

$$=\frac{27\pi}{5}$$

1976 AB4

(a) A point moves on the hyperbola $3x^2 - y^2 = 23$ so that its y-coordinate is increasing at a constant rate of 4 units per second. How fast is the x-coordinate changing when $x = 4$?

(b) For what value of k will the line $2x + 9y + k = 0$ be normal to the hyperbola $3x^2 - y^2 = 23$?

1976 AB4
Solution

(a) $3x^2 - y^2 = 23$

$6x\frac{dy}{dt} - 2y\frac{dy}{dt} = 0$

When $x = 4$, then $y = 5$ and so $(6)(4)\frac{dx}{dt} - 2(5)(4) = 0$

Therefore $\frac{dx}{dt} = \frac{5}{3}$.

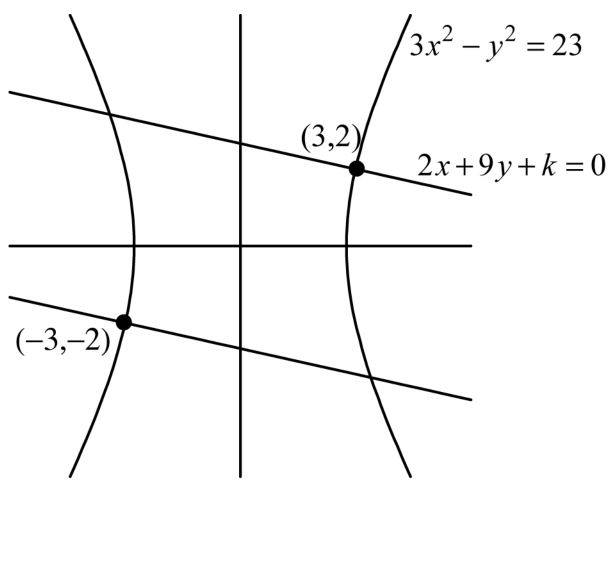
(b) The slope of line $2x + 9y + k = 0$ is $-\frac{2}{9}$. The slope of the normal will therefore be $\frac{9}{2}$.

$6x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$

$\frac{3x}{y} = \frac{9}{2} \Rightarrow y = \frac{2}{3}x \Rightarrow 3x^2 - \frac{4}{9}x^2 = 23 \Rightarrow x^2 = 9$

Therefore $x = \pm 3$. Since $y = \frac{2}{3}x$, the two points are $(3,2)$ and $(-3,-2)$.

At $(3,2)$, $k = -24$.
At $(-3,-2)$, $k = 24$.



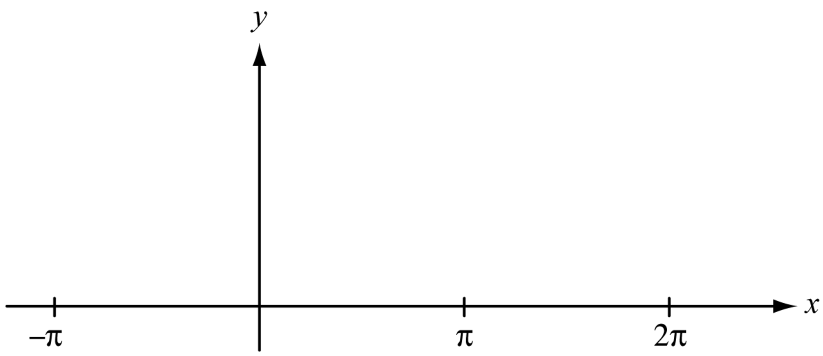
1976 AB5/BC3

Given the function defined by $y = e^{\sin x}$ for all x such that $-\pi \le x \le 2\pi$.

(a) Find the x- and y-coordinates of <u>all</u> maximum and minimum points on the given interval. Justify your answers.

(b) On the axes provided, sketch the graph of the function.

(c) Write an equation for the axis of symmetry of the graph.



1976 AB5/BC3
Solution

(a) $y = e^{\sin x}, -\pi \le x \le 2\pi$

$\frac{dy}{dx} = e^{\sin x}(\cos x)$ and $\frac{d^2y}{dx^2} = e^{\sin x}(\cos^2 x - \sin x)$

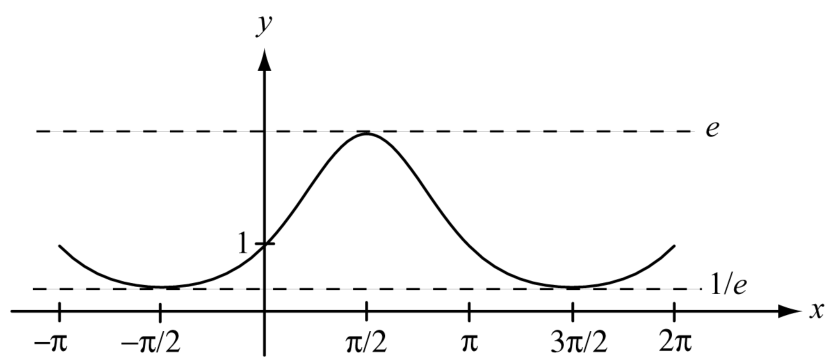
$\frac{dy}{dx} = 0$ when $\cos x = 0$. The critical points are at $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.

At both $x = -\frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, $\frac{d^2y}{dx^2} = \frac{1}{e} > 0$. At $x = \frac{\pi}{2}$, $\frac{d^2y}{dx^2} = -e < 0$.

Therefore there is a relative minimum at $\left(-\frac{\pi}{2}, \frac{1}{e}\right)$ and $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$, and a relative maximum at $\left(\frac{\pi}{2}, e\right)$. The endpoints are $(-\pi, 1)$ and $(2\pi, 1)$.

The maximum point is therefore at $\left(\frac{\pi}{2}, e\right)$ and the minimum points are at $\left(-\frac{\pi}{2}, \frac{1}{e}\right)$ and $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$.

(b)



(c) The axis of symmetry is the vertical line $x = \frac{\pi}{2}$

1976 AB6

(a) Given $5x^3 + 40 = \int_c^x f(t)\,dt$.

(i) Find $f(x)$.

(ii) Find the value of c.

(b) If $F(x) = \int_x^3 \sqrt{1+t^{16}}\,dt$, find $F'(x)$.

1976 AB6
Solution

(a) (i) Take the derivative of both sides of $5x^3 + 40 = \int_c^x f(t)\,dt$ to get

$15x^2 + 0 = f(x)$. Thus $f(x) = 15x^2$.

(ii) Method 1 (using the result from (i)):

$$5x^3 + 40 = \int_c^x 15t^2\,dt = 5t^3\Big|_c^x = 5x^3 - 5c^3$$

$$5c^3 = -40$$

$$c = -2$$

Method 2 (using the condition given in the stem with $x = c$):

$5c^3 + 40 = \int_0^0 f(t)dt = 0$ and so $c = -2$.

(b) $F(x) = \int_x^3 \sqrt{1+t^{16}}\,dt = -\int_3^x \sqrt{1+t^{16}}\,dt$

$$F'(x) = \frac{d}{dx}\left(-\int_3^x \sqrt{1+t^{16}}\,dt\right) = -\sqrt{1+x^{16}}$$

1976 AB7/BC6

For a differentiable function f, let f^* be the function defined by

$$f^*(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{h}.$$

(a) Determine $f^*(x)$ for $f(x) = x^2 + x$

(b) Determine $f^*(x)$ for $f(x) = \cos x$

(c) Write an equation that expresses the relationship between f^* and f', where f' denotes the usual derivative of f.

1976 AB7/BC6
Solution

(a) For $f(x) = x^2 + x$

$$f^*(x) = \lim_{h\to 0} \frac{\left((x+h)^2 + (x+h)\right) - \left((x-h)^2 + (x-h)\right)}{h}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 + x + h) - (x^2 - 2xh + h^2 + x - h)}{h}$$

$$= \lim_{h\to 0} \frac{4xh + 2h}{h}$$

$$= 4x + 2 = 2(2x+1)$$

(b) For $f(x) = \cos x$

$$f^*(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x-h)}{h}$$

$$= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - (\cos x \cos h + \sin x \sin h)}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin x \sin h}{h}$$

$$= -2\sin x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= -2\sin x \cdot 1 = -2\sin x$$

(c) $f^*(x) = 2f'(x)$

1976 BC1

A particle moves on the x-axis in such a way that its position at time t is given by $x(t) = (2t-1)(t-1)^2$.

(a) At what times t is the particle at rest? Justify your answer.

(b) During what interval of time is the particle moving to the left? Justify your answer.

(c) At what time during the interval found in part (b) is the particle moving most rapidly (that is, the speed is a maximum)? Justify your answer.

1976 BC1
Solution

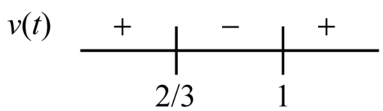
(a) $v(t) = x'(t) = 2(2t-1)(t-1) + 2(t-1)^2 = 2(t-1)(3t-2)$

The particle is at rest when $v(t) = 0$. This occurs when $t = \frac{2}{3}$ and $t = 1$.

(b) The motion is to the left when $v(t) < 0$.

$v(t) < 0$ when $t - 1 < 0$ and $3t - 2 > 0$, i.e. when $t < 1$ and $t > \frac{2}{3}$.

$v(t) < 0$ when $t - 1 > 0$ and $3t - 2 < 0$, i.e. when $t > 1$ and $t < \frac{2}{3}$. But this condition is impossible.



Hence the particle is moving to the left only for $\frac{2}{3} < t < 1$.

(c) Since the velocity is negative on the interval, the maximum speed will occur when the velocity is a minimum.

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 2(t-1)(3) + 2\left(3t-2\right)\left(1\right) = 12t - 10 = 12\left(t - \frac{5}{6}\right)$$

$a = 0$ when $t = \frac{5}{6}$.

$a < 0$ for $\frac{2}{3} < t < \frac{5}{6}$ and $a > 0$ for $\frac{5}{6} < t < 1$. Therefore the velocity is a minimum at $t = \frac{5}{6}$, and so this is the time that the speed is a maximum on the interval $\frac{2}{3} < t < 1$.

1976 BC4

(a) Determine $\int x^2 e^{5x} dx$.

(b) Using integration by parts, derive a general formula for $\int x^n e^{kx} dx$, $k \neq 0$, in which the resulting integrand involves x^{n-1}.

1976 BC4
Solution

(a) Let $u = x^2$, $du = 2xdx$, $dv = e^{5x}dx$ and $v = \frac{1}{5}e^{5x}$

$$\int x^2 e^{5x}\,dx = \frac{1}{5}x^2 e^{5x} - \frac{2}{5}\int xe^{5x}\,dx$$

Let $u = x$, $du = dx$, $dv = e^{5x}dx$, and $v = \frac{1}{5}e^{5x}$

$$\int x^2 e^{5x}\,dx = \frac{1}{5}x^2 e^{5x} - \frac{2}{5}\left(\frac{1}{5}xe^{5x} - \frac{1}{5}\int e^{5x}\,dx\right)$$

$$= \frac{1}{5}x^2 e^{5x} - \frac{2}{25}xe^{5x} + \frac{2}{125}e^{5x} + C$$

(b) Let $u = x^n$, $du = nx^{n-1}dx$, $dv = e^{kx}dx$, and $v = \frac{1}{k}e^{kx}$

$$\int x^n e^{kx}\,dx = \frac{1}{k}x^n e^{kx} - \frac{n}{k}\int x^{n-1}e^{kx}\,dx$$

1976 BC5

(a) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 3y = 0$.

(b) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 3y = 1 - e^{-x}$.

(c) Find the solution of the differential equation in (b) that satisfies the conditions that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$.

1976 BC5
Solution

(a) $y'' + 3y = 0$

$m^2 + 3 = 0$

$m = 0 \pm i\sqrt{3}$

The general solution is $y = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)$.

(b) Let $y = A + Be^{-x}$

$y' = -Be^{-x}$

$y'' = Be^{-x}$

$y'' + 3y = Be^{-x} + 3(A + Be^{-x}) = 3A + 4Be^{-x} = 1 - e^{-x}$

So $3A = 1$ and $4B = -1$. Hence $A = \frac{1}{3}$ and $B = -\frac{1}{4}$.

The general solution is $y = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x) + \frac{1}{3} - \frac{1}{4}e^{-x}$.

(c) $1 = y(0) = C_1 \cos(\sqrt{3} \cdot 0) + C_2 \sin(\sqrt{3} \cdot 0) + \frac{1}{3} - \frac{1}{4}e^{-0} = C_1 + \frac{1}{12}$

Hence $C_1 = \frac{11}{12}$.

$y'(x) = -\frac{11}{12}\sqrt{3}\sin(\sqrt{3}x) + C_2\sqrt{3}\cos(\sqrt{3}x) + 0 + \frac{1}{4}e^{-x}$

$0 = y'(0) = -\frac{11}{12}\sqrt{3}\sin(\sqrt{3} \cdot 0) + C_2\sqrt{3}\cos(\sqrt{3} \cdot 0) + \frac{1}{4}e^{-0} = C_2\sqrt{3} + \frac{1}{4}$

Hence $C_2 = -\frac{1}{4\sqrt{3}}$.

The solution is $y(x) = \frac{11}{12}\cos(\sqrt{3}x) - \frac{1}{4\sqrt{3}}\sin(\sqrt{3}x) + \frac{1}{3} - \frac{1}{4}e^{-x}$.

1976 BC7

(a) Write the first three nonzero terms and the general term of the Taylor series expansion about $x = 0$ of $f(x) = 5\sin\frac{x}{2}$.

(b) What is the interval of convergence for the series found in part (a)? Show your method.

(c) What is the minimum number of terms of the series in (a) that are necessary to approximate $f(x)$ on the interval $(-2, 2)$ with an error not exceeding 0.1? Show your method.

1976 BC7
Solution

(a)

$f(x)=5\sin\left(\frac{x}{2}\right)$	$f(0)=0$	$f'''(x)=\frac{-5}{2^3}\cos\left(\frac{x}{2}\right)$	$f'''(0)=\frac{-5}{2^3}$
$f'(x)=\frac{5}{2}\cos\left(\frac{x}{2}\right)$	$f'(0)=\frac{5}{2}$	$f^{(4)}(x)=\frac{5}{2^4}\sin\left(\frac{x}{2}\right)$	$f^{(4)}(0)=0$
$f''(x)=\frac{-5}{2^2}\sin\left(\frac{x}{2}\right)$	$f''(0)=0$	$f^{(5)}(x)=\frac{5}{2^5}\cos\left(\frac{x}{2}\right)$	$f^{(5)}(0)=\frac{5}{2^5}$

$$f(x)=\frac{5}{2}x-\frac{5}{2^3}\frac{x^3}{3!}+\frac{5}{2^5}\frac{x^5}{5!}+\cdots+(-1)^{n-1}\frac{5}{2^{2n-1}}\frac{x^{2n-1}}{(2n-1)!}+\cdots$$

(Note: one could also use the Taylor series for $\sin(x)$ as a starting point.)

(b) Ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{5}{2^{2n+1}(2n+1)!}x^{2n+1}}{\frac{5}{2^{2n-1}(2n-1)!}x^{2n-1}}\right|=\lim_{n\to\infty}\left|\frac{x^2}{2^2(2n+1)(2n)}\right|=0<1 \text{ for all } x$$

Therefore the series converges for all real numbers.

(Or, since the Taylor series for $\sin x$ converges for all x, so does the series for $f(x)=5\sin\left(\frac{x}{2}\right)$.)

(c) Two (nonzero) terms are sufficient for all x on the interval $|x|<2$.

$$f(x)=\frac{5}{2}x-\frac{5}{2^3}\frac{x^3}{3!}+0+R_5(x)$$

$$|R_5(x)|=\left|\frac{5\cos(c/2)}{2^5 5!}x^5\right|\le\frac{5}{2^5 5!}\cdot 2^5=\frac{5}{5!}<\frac{1}{10}$$

Two terms are necessary for the approximation to work for all x in the interval.

$f(1.9)=5\sin\frac{1.9}{2}<5\sin 1<5\sin\frac{\pi}{3}=5\cdot\frac{\sqrt{3}}{2}<5\cdot\frac{1.8}{2}=4.5$, but $\frac{5}{2}\cdot 1.9=4.75$.

Hence $\left|f(x)-\frac{5}{2}x\right|>0.25>0.1$ at $x=1.9$, so one term is not enough.

1977 AB1

Let $f(x) = \cos x$ for $0 \le x \le 2\pi$ and let $g(x) = \ln x$ for all $x > 0$. Let S be the composition of g with f, that is, $S(x) = g\left(f(x)\right)$.

(a) Find the domain of S.

(b) Find the range of S.

(c) Find the zeros of S.

(d) Find the slope of the line tangent to the graph of S at $x = \dfrac{\pi}{3}$.

1977 AB1
Solution

(a) $S(x) = g(f(x)) = \ln(\cos x)$

The domain of S is all x in the domain of f for which $\cos x > 0$, that is, $0 \le x < \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2} < x \le 2\pi$.

(b) The range of S is $y \le 0$.

(c) $\ln(\cos x) = 0$

$\cos x = e^0 = 1$

The zeros are $x = 0$ and $x = 2\pi$.

(d) $S'(x) = \dfrac{1}{\cos x}(-\sin x) = -\tan x$

The slope is $S'\left(\dfrac{\pi}{3}\right) = -\tan\dfrac{\pi}{3} = -\sqrt{3}$

1977 AB2

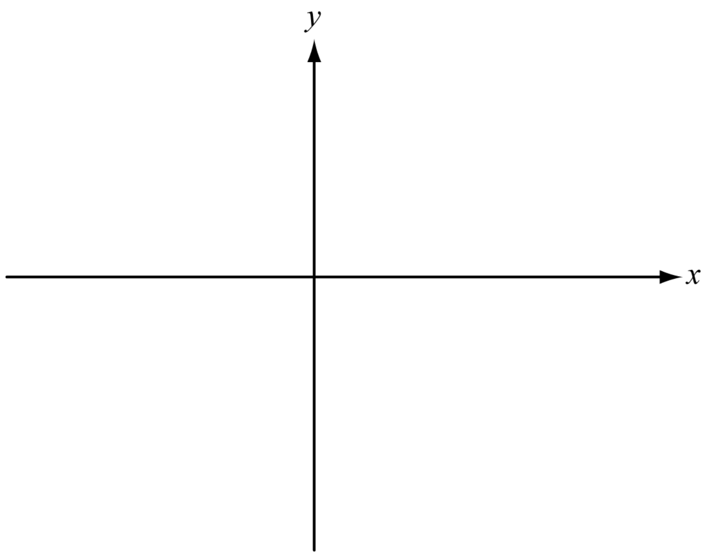
Consider the function f defined by $f(x) = (x^2 - 1)^3$ for all real numbers x.

(a) For what values of x is the function increasing?

(b) Find the x- and y-coordinates of the relative maximum and minimum points. Justify your answer.

(c) For what values of x is the graph of f concave upward?

(d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.



1977 AB2
Solution

(a) $f(x) = (x^2 - 1)^3$

$f'(x) = 6x(x^2 - 1)^2$

$x < 0 \Rightarrow f'(x) < 0$

$x > 0 \Rightarrow f'(x) > 0$

Therefore the function is increasing for $x > 0$.

(b) $f'(x) = 6x(x^2 - 1)^2 = 0$

$x = 0, x = 1, x = -1$

Since f is decreasing for $x < 0$ and increasing for $x > 0$, the only relative minimum point is at $x = 0, y = -1$ and there are no relative maximum points.

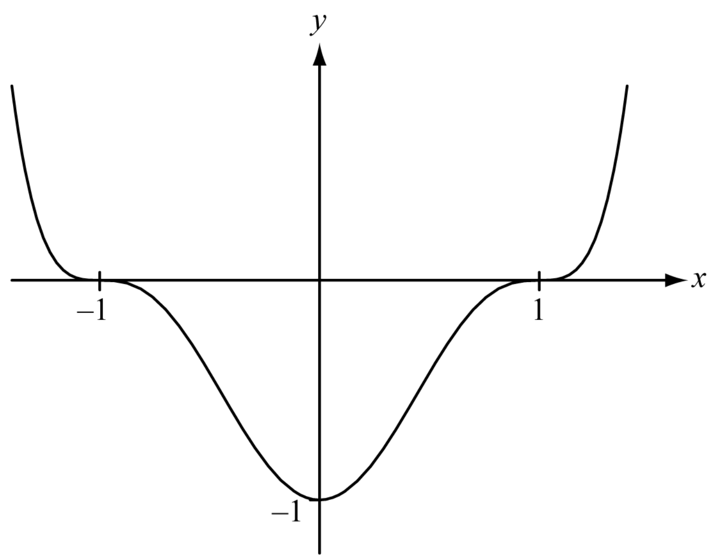
(c) $f''(x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1) = 6(x^2 - 1)(5x^2 - 1)$

$f''(x) = 0$ for $x = 1, x = -1, x = \sqrt{\frac{1}{5}}, x = -\sqrt{\frac{1}{5}}$.

The graph of f is concave up when $6(x^2 - 1)(5x^2 - 1) > 0$. This happens for all x in

$(-\infty, -1) \cup \left(-\sqrt{\frac{1}{5}}, \sqrt{\frac{1}{5}}\right) \cup (1, \infty)$

(d)



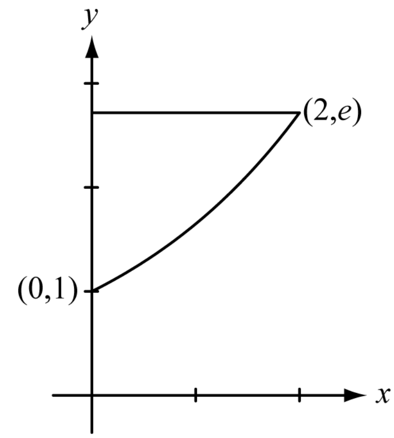
1977 AB3

Given the function f defined for all real numbers x by $f(x) = e^{x/2}$.

(a) Find the area of the region R bounded by the line $y = e$, the graph of f, and the y-axis.

(b) Find the volume of the solid generated by revolving R, the region in part (a), about the x-axis.

1977 AB3
Solution



(a) Area $= \int_0^2 (e - y)\,dx = \int_0^2 (e - e^{x/2})\,dx$

$$= (ex - 2x^{x/2})\Big|_0^2 = ((2e - 2e) - (0 - 2)) = 2$$

or

Area $= \int_1^e x\,dy = 2\int_1^e \ln y\,dy$

$$= 2\,(y \ln y - y)\Big|_1^e = 2((e - e) - (0 - 1)) = 2$$

(b) <u>Disks</u>:

Volume $= \pi\int_0^2 (e^2 - e^x)\,dx$

$$= \pi(e^2 x - e^x)\Big|_0^2 = \pi((2e^2 - e^2) - (0 - 1)) = \pi(e^2 + 1)$$

<u>Shells</u>:

Volume $= 2\pi\int_1^e xy\,dy = 2\pi\int_1^e 2y \ln y\,dy = 4\pi\int_1^e y \ln y\,dy$

$$= 4\pi\left(\frac{y^2}{2}\ln y - \frac{y^2}{4}\right)\Bigg|_1^e$$

$$= 4\pi\left(\left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \left(0 - \frac{1}{4}\right)\right) = 4\pi\left(\frac{e^2}{4} + \frac{1}{4}\right) = \pi(e^2 + 1)$$

1977 AB4/BC2

Let f and g and their inverses f^{-1} and g^{-1} be differentiable functions and let the values of $f,\ g,$ and the derivatives f' and g' at $x=1$ and $x=2$ be given by the table below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	5	4
2	2	π	6	7

Determine the value of each of the following.

(a) The derivative of $f+g$ at $x=2$

(b) The derivative of fg at $x=2$

(c) The derivative of $\dfrac{f}{g}$ at $x=2$

(d) $h'(1)$ where $h(x)=f(g(x))$

(e) The derivative of g^{-1} at $x=2$

1977 AB4/BC2
Solution

(a) $(f+g)'(2) = f'(2) + g'(2) = 6 + 7 = 13$

(b) $(fg)'(2) = f(2)g'(2) + f'(2)g(2) = 2 \cdot 7 + 6 \cdot \pi = 14 + 6\pi$

(c) $\left(\dfrac{f}{g}\right)'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{\pi \cdot 6 - 2 \cdot 7}{\pi^2} = \dfrac{6\pi - 14}{\pi^2}$

(d) $(f \circ g)'(1) = f'(g(1))g'(1) = f'(2) \cdot 4 = 6 \cdot 4 = 24$

(e) $h = g^{-1}(g(h))$. Therefore $1 = (g^{-1})'(g(h)) \cdot g'(h)$. Let $h = 1$. Then

$1 = (g^{-1})'(2) \cdot g'(1)$

$(g^{-1})'(2) = \dfrac{1}{g'(1)} = \dfrac{1}{4}$

1977 AB5/BC3

A particle moves along the x-axis with acceleration given by $a(t) = 2t - 10 + \frac{12}{t}$ for $t \geq 1$.

(a) Write an expression for the velocity $v(t)$, given that $v(1) = 9$.

(b) For what values of t, $1 \leq t \leq 3$, is the velocity a maximum? Justify your answer.

(c) Write an expression for the position $x(t)$, given that $x(1) = -16$.

1977 AB5/BC3
Solution

(a) $v(t) = t^2 - 10t + 12\ln t + C$

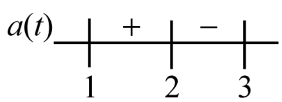
$9 = v(1) = 1 - 10 + 12(0) + C$

$C = 18$

$v(t) = t^2 - 10t + 12\ln t + 18$

(b) $a(t) = \dfrac{2t^2 - 10t + 12}{t} = \dfrac{2(t-2)(t-3)}{t}$

$a(t) = 0$ when $t = 2$ and $t = 3$.



Since the velocity is increasing for $1 \le t < 2$ and decreasing for $2 < t < 3$, the velocity is a maximum at $t = 2$.

or

The maximum will be at $t = 2$ or at an endpoint.

$v''(t) = a'(t) = \dfrac{2t^2 - 12}{t^2}$.

Since $v'(3) = 0$ and $v''(3) = \dfrac{2}{3} > 0$, then v has a relative minimum at $t = 3$.

$v(1) = 9 < v(2) = 2 + 12\ln 2 \approx 2 + 12 \cdot (0.7) = 10.4$

Therefore the maximum velocity is at $t = 2$.

(c) $x(t) = \int v(t)\,dt = \dfrac{t^3}{3} - 5t^2 + 18t + 12(t\ln t - t) + D$

$-16 = \dfrac{1}{3} - 5 + 18 - 12 + D$

$D = -16 - \dfrac{4}{3} = -\dfrac{52}{3}$

$x(t) = \dfrac{t^3}{3} - 5t^2 + 6t + 12t\ln t - \dfrac{52}{3}$

1977 AB6

A rectangle has a constant area of 200 square meters and its length L is increasing at the rate of 4 meters per second.

(a) Find the width W at the instant the width is decreasing at the rate of 0.5 meters per second.

(b) At what rate is the diagonal D of the rectangle changing at the instant when the width W is 10 meters?

1977 AB6
Solution

(a) Implicit

$$LW = 200$$

$$L\frac{dW}{dt} + W\frac{dL}{dt} = 0$$

$$(-0.5)L + 4W = 0$$
$$L = 8W$$
$$8W^2 = 200$$

$$W = \sqrt{\frac{200}{8}} = \sqrt{25} = 5$$

Explicit

$$W = \frac{200}{L}$$

$$\frac{dW}{dt} = -\frac{200}{L^2}\frac{dL}{dt}$$

$$-0.5 = -\frac{200}{L^2}\cdot 4$$
$L^2 = 1600$ so $L = 40$

$$W = \frac{200}{40} = 5$$

(b) Implicit with W

$$D^2 = L^2 + W^2$$

When $W = 10$, $L = 20$ and $D^2 = 500$

$$2D\frac{dD}{dt} = 2L\frac{dL}{dt} + 2W\frac{dW}{dt}$$

From (a),
$$\frac{dW}{dt} = -\frac{W}{L}\cdot\frac{dL}{dt} = -\frac{10}{20}\cdot 4 = -2$$

$$\frac{dD}{dt} = \frac{1}{10\sqrt{5}}\left(20\cdot 4 + 10(-2)\right)$$

$$\frac{dD}{dt} = \frac{6}{\sqrt{5}}$$

Explicit

$$D = \sqrt{L^2 + W^2}$$

$$D = \sqrt{L^2 + \left(\frac{200}{L}\right)^2}$$

$$\frac{dD}{dt} = \frac{2L - \frac{2(200)^2}{L^3}}{2\sqrt{L^2 + \left(\frac{200}{L}\right)^2}}\cdot\frac{dL}{dt}$$

$$\frac{dD}{dt} = \frac{2\cdot 20 - 10}{2\sqrt{20^2 + 10^2}}\cdot 4$$

$$\frac{dD}{dt} = \frac{6}{\sqrt{5}}$$

1977 AB7/BC6

Let f be the real-valued function defined by $f(x) = \sin^3 x + \sin^3 |x|$.

(a) Find $f'(x)$ for $x > 0$.

(b) Find $f'(x)$ for $x < 0$.

(c) Determine whether $f(x)$ is continuous at $x = 0$. Justify your answer.

(d) Determine whether the derivative of $f(x)$ exists at $x = 0$. Justify your answer.

1977 AB7/BC6
Solution

(a) For $x > 0$

$$f(x) = \sin^3 x + \sin^3 x = 2\sin^3 x$$

$$f'(x) = 6\sin^2 x \cos x$$

(b) For $x < 0$

$$f(x) = \sin^3 x + \sin^3(-x) = \sin^3 x - \sin^3 x = 0$$

$$f'(x) = 0$$

(c) $f(0) = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2\sin^3 x = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$$

Since $\lim\limits_{x \to 0} f(x) = 0 = f(0)$, the function f is continuous at $x = 0$.

(d) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ if the limit exists. At $x = 0$

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{2\sin^3 h}{h} = \lim_{h \to 0^+} 2 \cdot \sin^2 h \cdot \frac{\sin h}{h} = 2 \cdot 0 \cdot 1 = 0$$

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{0}{h} = \lim_{h \to 0^-} 0 = 0$$

Therefore $\lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h} = 0$ and so $f'(0)$ exists and equals 0.

1977 BC1

Let f be the function defined by $f(x) = \dfrac{x}{1+x^2}$ for all real numbers x.

(a) (i) Find the zeros of f

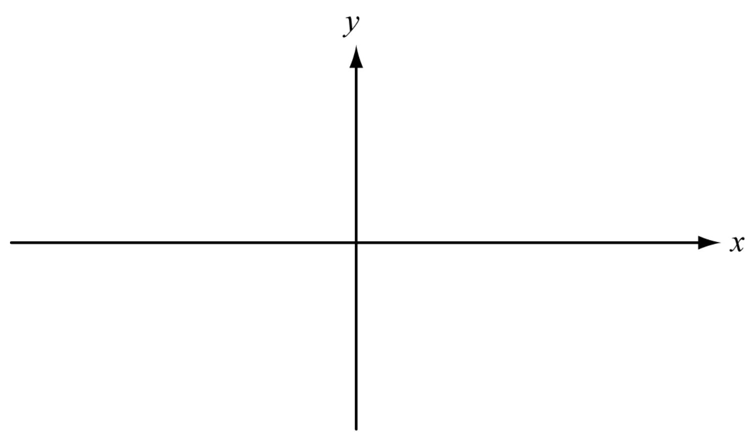
(ii) Investigate the symmetry of the graph of f with respect to the x-axis, y-axis, and the origin.

(iii) Find the x- and y- coordinates of the relative maximum and minimum points of f. Justify your answer.

(iv) Describe the behavior of the graph of f for large $|x|$.

(v) Using the information found above, sketch the graph of f on the axes provided.

(b) Find the area of the region bounded by the graph of f, the x-axis, and the lines $x = \dfrac{1}{k}$ and $x = k$ for $k > 1$.



1977 BC1
Solution

(a) (i) $f(x) = \dfrac{x}{1+x^2} = 0$ only when $x = 0$.

(ii) $f(-x) = \dfrac{-x}{1+(-x)^2} = -\dfrac{x}{1+x^2} = -f(x)$. Therefore the graph is symmetric about the origin. There are no other symmetries.

(iii) $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$. So $f'(x) = 0$ when $x = 1$ and $x = -1$.

$f''(x) = \dfrac{-6x+2x^3}{(1+x^2)^3}$

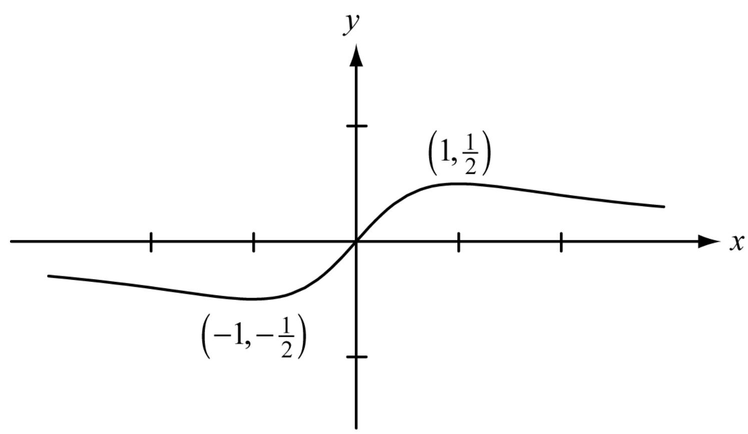
Since $f''(-1) = \dfrac{4}{8} > 0$, there is a relative minimum at $\left(-1, -\dfrac{1}{2}\right)$.

Since $f''(1) = \dfrac{-4}{8} < 0$, there is a relative maximum at $\left(1, \dfrac{1}{2}\right)$.

Alternatively, $f'(x) > 0$ for $|x| < 1$ and $f'(x) < 0$ for $|x| > 1$. Therefore there is a relative minimum at $\left(-1, -\dfrac{1}{2}\right)$ and a relative maximum at $\left(1, \dfrac{1}{2}\right)$.

(iv) As $|x| \to \infty$, $f(x) \to 0$. So the graph approaches the x-axis asymptotically.

(v)



(b) Area $= \displaystyle\int_{\frac{1}{k}}^{k} \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2)\Big|_{\frac{1}{k}}^{k} = \frac{1}{2}\ln(1+k^2) - \frac{1}{2}\ln\left(1+\frac{1}{k^2}\right)$

$= \dfrac{1}{2}\ln(1+k^2) - \left(\dfrac{1}{2}\ln(k^2+1) - \dfrac{1}{2}\ln(k^2)\right) = \dfrac{1}{2}\ln(k^2) = \ln k$

1977 BC4

(a) If $\sqrt{2}e^{2x}$ is a particular solution of the differential equation $y'' - 4y' + py = 0$, find the value of p and the general solution of the equation.

(b) Find the particular solution of the differential equation $y'' - 4y' + 5y = 0$ that satisfies the condition that $y = 0$ and $y' = 1$ when $x = \frac{\pi}{2}$.

1977 BC4
Solution

(a) Method 1:

$y = \sqrt{2}e^{2x}$

$y' = 2\sqrt{2}e^{2x}$

$y'' = 4\sqrt{2}e^{2x}$

Substituting into the differential equation

$(4\sqrt{2} - 8\sqrt{2} + p\sqrt{2})e^{2x} = 0$

Therefore $p = 4$.

The differential equation is now
$y'' - 4y' + 4y = 0$
The auxiliary equation is
$k^2 - 4k + 4 = (k-2)^2 = 0$
whose only solution is $k = 2$. Therefore the general solution to the differential equation is
$y = (c_1 + c_2 x)e^{2x}$.

Method 2:
The auxiliary equation is
$k^2 - 4k + p = 0$

The roots are

$k = \dfrac{4 \pm \sqrt{16 - 4p}}{2} = 2 \pm \sqrt{4 - p}$

For $\sqrt{2}e^{2x}$ to be a solution to the differential equation, one root must be 2. Therefore $p = 4$ and $k = 2$ is a double root.

The general solution is therefore
$y = (c_1 + c_2 x)e^{2x}$

(b) The auxiliary equation for $y'' - 4y' + 5y = 0$ is
$k^2 - 4k + 5 = 0$

$k = \dfrac{4 \pm \sqrt{-4}}{2} = 2 \pm i$

Because the roots are complex, the general solution is $y = e^{2x}(c_1 \cos x + c_2 \sin x)$.

At $x = \dfrac{\pi}{2}$, $0 = y = c_1 \cdot 0 + c_2 \cdot 1 = c_2$.

$y = e^{2x}(c_1 \cos x)$

$y' = e^{2x}(2c_1 \cos x - c_1 \sin x)$

At $x = \dfrac{\pi}{2}$, $1 = y' = e^{\pi}(2c_1 \cdot 0 - c_1 \cdot 1)$ and so $c_1 = -e^{-\pi}$.

The solution is $y = -e^{2x-\pi} \cos x$.

1977 BC5

(a) Does the series $\sum_{j=1}^{\infty} \frac{2}{j^2} \sin\left(\frac{\pi}{j}\right)$ converge? Justify your answer.

(b) Express $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n} \sin\left(\frac{k\pi}{n}\right)$ as a definite integral and evaluate the integral.

1977 BC5
Solution

(a) Comparison Test

$$0 \le \frac{2}{j^2}\sin\left(\frac{\pi}{j}\right) \le \frac{2}{j^2} \text{ for } j \ge 1$$

The series $\sum_{j=1}^{\infty}\frac{2}{j^2}$ converges (p-series for $p > 1$), so the series $\sum_{j=1}^{\infty}\frac{2}{j^2}\sin\left(\frac{\pi}{j}\right)$ also converges.

Note: Could also use the integral test or show that the series converges absolutely.

(b) The finite sum $\sum_{k=1}^{n}\frac{2}{n}\sin\left(\frac{k\pi}{n}\right)$ is a right Riemann sum for the function $f(x) = 2\sin(\pi x)$ over the interval $0 \le x \le 1$. Therefore

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\sin\left(\frac{k\pi}{n}\right) &= \int_0^1 2\sin(\pi x)\,dx \\ &= -\frac{2}{\pi}\cos(\pi x)\Big|_0^1 = -\frac{2}{\pi}(\cos\pi - \cos 0) \\ &= -\frac{2}{\pi}(-1-1) = \frac{4}{\pi}\end{aligned}$$

1977 BC7

Let $F(x) = \int_0^x \frac{1}{1+t^4}\,dt$ for all real numbers x.

(a) Find $F(0)$.

(b) Find $F'(1)$.

(c) Justify that $F(3) - F(1) < 1$.

(d) Justify that $F(x) + F(-x) = 0$ for all real numbers x.

1977 BC7
Solution

(a) $F(0)=\int_0^0 \frac{1}{1+t^4}=0$

(b) $F'(x)=\frac{1}{1+x^4}$

$F'(1)=\frac{1}{2}$

(c) $F(3)-F(1)=\int_1^3 \frac{1}{1+t^4}\,dt$

We have $\frac{1}{1+t^4}\le\frac{1}{2}$ for $1\le t\le 3$. The inequality is strict except at one point. Hence

$$F(3)-F(1)=\int_1^3 \frac{1}{1+t^4}\,dt<\int_1^3 \frac{1}{2}\,dt=2\left(\frac{1}{2}\right)=1$$

(d) $\int_{-x}^0 \frac{1}{1+t^4}\,dt=\int_0^x \frac{1}{1+t^4}\,dt$ since $\frac{1}{1+t^4}$ is an even function. Therefore

$$\begin{aligned} F(x)+F(-x) &= \int_0^x \frac{1}{1+t^4}\,dt+\int_0^{-x} \frac{1}{1+t^4}\,dt \\ &= \int_0^x \frac{1}{1+t^4}\,dt-\int_{-x}^0 \frac{1}{1+t^4}\,dt \\ &= \int_0^x \frac{1}{1+t^4}\,dt-\int_0^x \frac{1}{1+t^4}\,dt \\ &= 0 \end{aligned}$$

1978 AB1

Given the function f defined by $f(x) = x^3 - x^2 - 4x + 4$.

(a) Find the zeros of f.

(b) Write an equation of the line tangent to the graph of f at $x = -1$.

(c) The point (a, b) is on the graph of f and the line tangent to the graph at (a, b) passes through the point $(0, -8)$, which is not on the graph of f. Find the values of a and b.

1978 AB1
Solution

(a) The zeros of $f(x) = x^3 - x^2 - 4x + 4$ can be found by factoring. The rational candidates are $\pm 1, \pm 2, \pm 4$. Since $f(1) = 0$, long (or synthetic) division gives

$$\begin{array}{r|rrrr} 1 & 1 & -1 & -4 & 4 \\ & & 1 & 0 & -4 \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

Therefore $f(x) = (x-1)(x^2-4) = (x-1)(x-2)(x+2)$ and so the zeros are $x = 1, 2, -2$.

(b) $f'(x) = 3x^2 - 2x - 4$

$f'(-1) = 3(-1)^2 - 2(-1) - 4 = 1$

$f(-1) = (-1)^3 - (-1)^2 - 4(-1) + 4 = 6$

The tangent line is $y - 6 = (1)(x+1)$ or $y = x + 7$.

(c) The slope of the line from (a, b) to $(0, -8)$ is equal to $f'(a)$. This gives

$$\frac{b+8}{a} = 3a^2 - 2a - 4$$

Since (a,b) is on the graph, we also have

$$b = a^3 - a^2 - 4a + 4.$$

Combining the two equations gives

$$3a^3 - 2a^2 - 4a - 8 = a^3 - a^2 - 4a + 4$$
$$2a^3 - a^2 - 12 = 0$$
$$a = 2$$
$$b = 8 - 4 - 8 + 4 = 0$$

1978 AB2

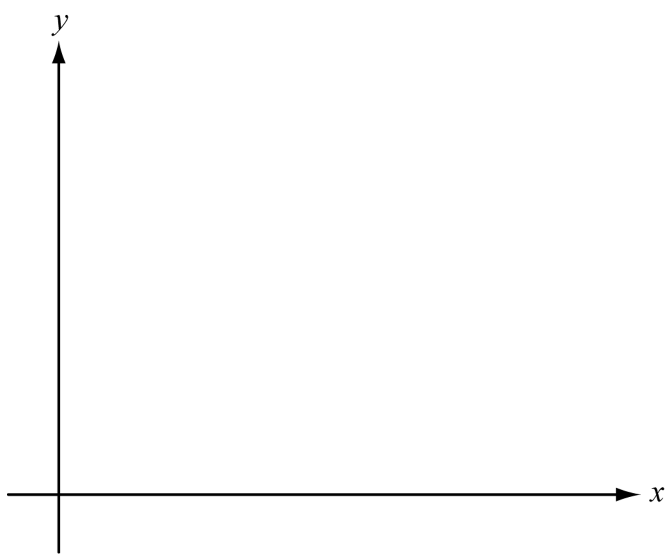
Let $f(x)=(1-x)^2$ for all real numbers x, and let $g(x)=\ln x$ for all $x>0$. Let $h(x)=(1-\ln x)^2$.

(a) Determine whether $h(x)$ is the composition $f(g(x))$ or the composition $g(f(x))$.

(b) Find $h'(x)$.

(c) Find $h''(x)$.

(d) On the axes provided, sketch the graph of h.



1978 AB2
Solution

(a) $f(g(x)) = f(\ln x) = (1-\ln x)^2$

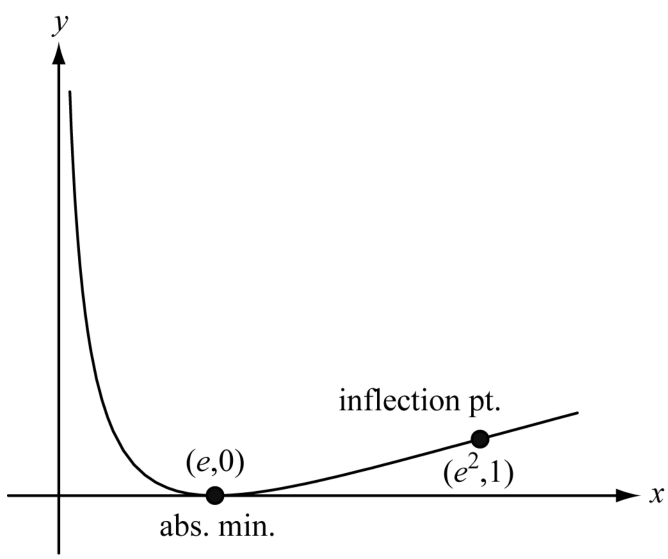
$g(f(x)) = g((1-x)^2) = \ln((1-x)^2)$

Therefore $h(x) = f(g(x))$.

(b) $h'(x) = 2(1-\ln x)\left(-\frac{1}{x}\right) = 2\cdot\frac{\ln x - 1}{x}$

(c) $h''(x) = 2\cdot\frac{x\cdot\frac{1}{x}-(\ln x - 1)}{x^2} = 2\cdot\frac{2-\ln x}{x^2}$

(d)



1978 AB3

Given the function f defined by $f(x) = \dfrac{2x-2}{x^2+x-2}$.

(a) For what values of x is $f(x)$ discontinuous?

(b) At each point of discontinuity found in part (a), determine whether $f(x)$ has a limit and, if so, give the value of the limit.

(c) Write an equation for each vertical and horizontal asymptote to the graph of f. Justify your answer.

(d) A rational function $g(x) = \dfrac{a}{b+x}$ is such that $g(x) = f(x)$ wherever f is defined. Find the values of a and b.

1978 AB3
Solution

(a) $f(x) = \dfrac{2(x-1)}{(x-1)(x+2)}$

f is discontinuous when $(x-1)(x+2) = 0$. This occurs at $x = 1$ and $x = -2$.

(b) $\dfrac{2(x-1)}{(x-1)(x+2)} = \dfrac{2}{x+2}$ for $x \neq 1$.

At $x = -2$: $\lim\limits_{x \to -2} \dfrac{2}{x+2}$ does not exist

At $x = 1$: $\lim\limits_{x \to 1} \dfrac{2}{x+2}$ exists and has the value $\dfrac{2}{3}$

(c) The horizontal asymptote is $y = 0$ because $\lim\limits_{x \to \infty} \dfrac{2}{x+2} = 0$, or because $\lim\limits_{x \to -\infty} \dfrac{2}{x+2} = \lim\limits_{x \to +\infty} \dfrac{2}{x+2} = 0$.

The vertical asymptote is $x = -2$ because $\lim\limits_{x \to -2} \dfrac{2}{x+2} = \infty$.

(d) $f(x) = \dfrac{2}{x+2}$ for $x \neq 1$. Therefore $g(x) = \dfrac{2}{2+x}$ and so $a = 2$ and $b = 2$.

1978 AB4

A particle moves on the x-axis so that its velocity at any time t is given by $v(t) = \sin 2t$. At $t = 0$, the particle is at the origin.

(a) For $0 \le t \le \pi$, find all values of t for which the particle is moving to the left.

(b) Write an expression for the position of the particle at any time t.

(c) For $0 \le t \le \frac{\pi}{2}$, find the average value of the position function determined in part (b).

1978 AB4
Solution

(a) $v(t) = \sin 2t \quad 0 \le t \le \pi$

The particle is moving to the left when $\sin 2t < 0$. This happens when $\pi < 2t < 2\pi$, or $\frac{\pi}{2} < t < \pi$.

(b) $s(t) = \int \sin 2t \, dt = -\frac{1}{2}\cos 2t + C$

$0 = s(0) = -\frac{1}{2}\cos 0 + C = -\frac{1}{2} + C$

$C = \frac{1}{2}$

Therefore $s(t) = -\frac{1}{2}\cos 2t + \frac{1}{2}$

(c) Average Value $= \dfrac{1}{\frac{\pi}{2} - 0}\int_0^{\pi/2} -\frac{1}{2}(\cos 2t - 1)\, dt$

$$= -\frac{1}{\pi}\int_0^{\pi/2} (\cos 2t - 1)\, dt$$

$$= -\frac{1}{\pi}\left(\frac{1}{2}\sin 2t - t\right)\Bigg|_0^{\pi/2}$$

$$= -\frac{1}{\pi}\left(\frac{1}{2}\sin \pi - \frac{\pi}{2}\right)$$

$$= -\frac{1}{\pi}\left(0 - \frac{\pi}{2}\right) = \frac{1}{2}$$

1978 AB5/BC1

Given the curve $x^2 - xy + y^2 = 9$.

(a) Write a general expression for the slope of the curve.

(b) Find the coordinates of the points on the curve where the tangents are vertical.

(c) At the point $(0,3)$ find the rate of change in the slope of the curve with respect to x.

1978 AB5/BC1
Solution

(a) Implicit differentiation gives

$$2x - xy' - y + 2yy' = 0$$

$$(2y - x)y' = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

(b) There is a vertical tangent when $2y - x = 0$, so $x = 2y$. Substituting into the equation of the curve gives $(2y)^2 - (2y)y + y^2 = 9$, or $3y^2 = 9$. Therefore $y = \pm\sqrt{3}$ and the two points on the curve where the tangents are vertical are $(2\sqrt{3}, \sqrt{3})$ and $(-2\sqrt{3}, -\sqrt{3})$.

(c) $$y'' = \frac{(2y - x)(y' - 2) - (y - 2x)(2y' - 1)}{(2y - x)^2}$$

At the point $(0,3)$, $y' = \frac{3-0}{6-0} = \frac{1}{2}$ and so $y'' = \frac{(6-0)\left(\frac{1}{2} - 2\right) - (3-0)(1-1)}{(6-0)^2} = -\frac{1}{4}$

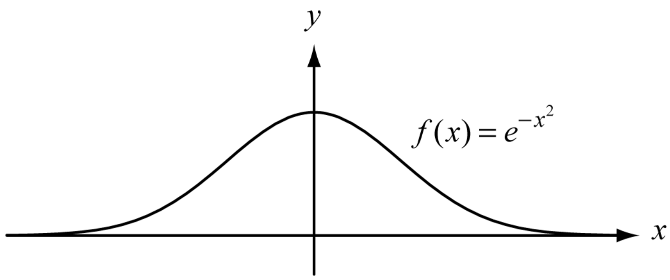
Alternatively, one can use implicit differentiation a second time to get

$$2 - xy'' - y' - y' + 2yy' + 2(y')^2 = 0$$

Substituting $x = 0,\ \ y = 3,$ and $y' = \frac{1}{2}$ gives

$$2 - 0 - \frac{1}{2} - \frac{1}{2} + 6y'' + 2\left(\frac{1}{4}\right) = 0 \Rightarrow 6y'' = -\frac{3}{2} \Rightarrow y'' = -\frac{1}{4}$$

1978 AB6/BC3



Given the function f defined by $f(x) = e^{-x^2}$.

(a) Find the maximum area of a rectangle that has two vertices on the x-axis and two on the graph of f. Justify your answer.

(b) Let R be the region in the first quadrant bounded by the x- and y-axes, the graph of f, and the line $x = k$. Find the volume of the solid generated by revolving R about the <u>y-axis</u>.

(c) Evaluate the limit of the volume determined in part (b) as k increases without bound.

1978 AB6/BC3
Solution

(a) $A(x) = 2xe^{-x^2}$

$A'(x) = 2e^{-x^2} + 2xe^{-x^2}(-2x) = 2e^{-x^2}(1-2x^2)$

$A'(x) = 0 \Rightarrow 1-2x^2 = 0$

$x = \frac{1}{\sqrt{2}}$

Since $A'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, the maximum area is

$A_{\text{max}} = 2\left(\frac{1}{\sqrt{2}}\right)e^{-1/2} = \sqrt{\frac{2}{e}}$

(b) Volume $= 2\pi\int_0^k xe^{-x^2}\,dx$

$= -\pi\int_0^k e^{-x^2}(-2x)\,dx = -\pi e^{-x^2}\Big|_0^k = -\pi\left(e^{-k^2} - 1\right) = \pi\left(1 - e^{-k^2}\right)$

or

Volume $= \pi\int_{e^{-k^2}}^1 \left(-\ln y\right)dy + \pi e^{-k^2} \cdot k^2$

$= -\pi\left(y\ln y - y\right)\Big|_{e^{-k^2}}^1 + \pi k^2 e^{-k^2} = \pi - \left(-\pi e^{-k^2}\ln e^{-k^2} + \pi e^{-k^2}\right) + \pi k^2 e^{-k^2}$

$= \pi - \pi e^{-k^2} = \pi\left(1 - e^{-k^2}\right)$

(c) $\lim_{k\to\infty} \pi\left(1 - e^{-k^2}\right) = \pi$

1978 AB7/BC6

Let g and h be any two twice-differentiable functions that are defined for all real numbers and that satisfy the following properties for all x:

(i) $(g(x))^2 + (h(x))^2 = 1$

(ii) $g'(x) = (h(x))^2$

(iii) $h(x) > 0$

(iv) $g(0) = 0$

(a) Justify that $h'(x) = -g(x)h(x)$ for all x.

(b) Justify that h has a relative maximum at $x = 0$.

(c) Justify that the graph of g has a point of inflection at $x = 0$.

1978 AB7/BC6
Solution

(a) $2g(x)g'(x)+2h(x)h'(x)=0$ from differentiating both sides of (i)

$g(x)(h(x))^2+h(x)h'(x)=0$ from (ii)

Since $h(x)\neq 0$ by (iii), we must have $g(x)h(x)+h'(x)=0$.

Therefore $h'(x)=-g(x)h(x)$

(b) Since $g(0)=0$, $h'(0)=-g(0)h(0)=0$.

$h''(x)=-g(x)h'(x)-g'(x)h(x)$

$h''(0)=-g(0)h'(0)-g'(0)h(0)=0-h(0)^2h(0)=-h(0)^3<0$ since $h(x)>0$ for all x.

Therefore h has a relative maximum at $x = 0$.

Alternatively, since $g'(x)=(h(x))^2$, g is an increasing function. Since $g(0)=0$, we must have that $g(x)<0$ for $x < 0$ and $g(x)>0$ for $x > 0$. Now $h'(x)=-g(x)h(x)$ and $h(x)>0$. Therefore $h'(x)>0$ for $x < 0$ and $h'(x)<0$ for $x > 0$. Hence h has a relative maximum at $x = 0$.

(c) $g''(x)=2h(x)h'(x)=-2g(x)(h(x))^2$. Therefore $g''(x)$ changes sign at $x = 0$ because $h(x)\neq 0$ and $g(x)$ changes sign at $x = 0$ (see part (b)). This implies that the graph of g has a point of inflection at $x = 0$.

1978 BC2

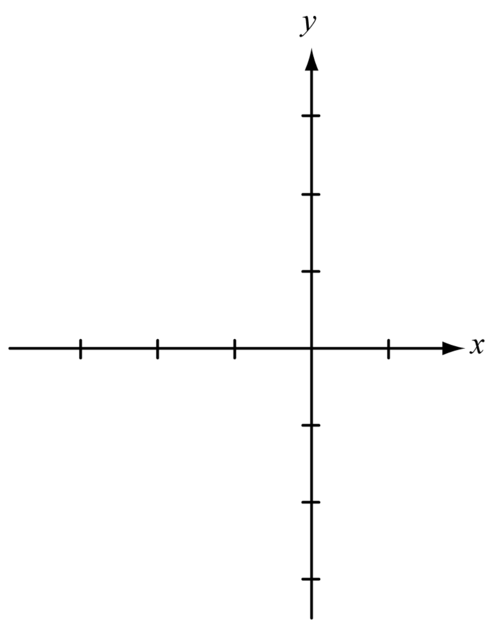
Given the function f defined by $f(x) = x|x+2|$ for all x such that $-3 \le x \le 1$.

(a) Find the values of x in the given interval for which f is increasing. Justify your answer.

(b) For what values of x is the graph of f concave downward?

(c) On the axes provided below, sketch the graph of f.

(d) Is $f'(x)$ continuous for all x in the given interval? Justify your answer.



1978 BC2
Solution

(a) $f(x)=\begin{cases}-x^2-2x & \text{for } -3\le x<-2\\ x^2+2x & \text{for } -2\le x\le 1\end{cases}$

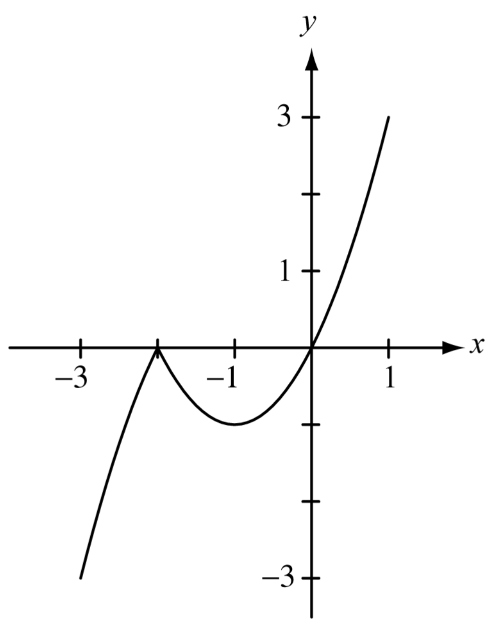
$f'(x)=\begin{cases}-2x-2 & \text{for } -3\le x<-2\\ 2x+2 & \text{for } -2< x\le 1\end{cases}$

One the interval $-3\le x<-2$, $f'(x)=-2x-2>0$ when $x<-1$, that is, for all x in this interval. On the interval $-2<x\le 1$, $f'(x)=2x+2>0$ when $x>-1$. Therefore f is increasing on the intervals $[-3,2)$ and $(-1,1]$.

(b) $f''(x)=\begin{cases}-2 & \text{for } -3\le x<-2\\ 2 & \text{for } -2< x\le 1\end{cases}$

The graph of f is concave downward on $[-3,-2)$.

(c)



(d) $\lim\limits_{x\to -2^+} f'(x)=-2$

$\lim\limits_{x\to -2^-} f'(x)=2$

Therefore $\lim\limits_{x\to -2} f'(x)$ does not exist and hence f' is not continuous at $x=-2$.

1978 BC4

(a) In the interval $0 < x < \frac{\pi}{2}$, find the general solution of the differential equation $(\cot x)\frac{dy}{dx} + y = \csc x$.

(b) Find the solution of the differential equation in part (a) that satisfies the condition that $y = 0$ when $x = \frac{\pi}{3}$.

1978 BC4
Solution

(a) $\frac{dy}{dx} + (\tan x)y = \sec x$

The integrating factor is $e^{\int \tan x\,dx} = e^{-\ln \cos x} = \sec x$. Multiplying by this factor gives

$(\sec x)\frac{dy}{dx} + (\sec x \tan x)y = \sec^2 x$

$\frac{d}{dx}((\sec x)y) = \sec^2 x$

$(\sec x)y = \tan x + C$

The general solution is $y = \sin x + C\cos x$.

(b) $0 = \sin\frac{\pi}{3} + C\cos\frac{\pi}{3}$

$0 = \frac{\sqrt{3}}{2} + \frac{1}{2}C$

$C = -\sqrt{3}$

The solution is $y = \sin x - \sqrt{3}\cos x$.

1978 BC5

The power series $\sum_{n=0}^{\infty} \frac{\ln(n+1)}{n+1} x^n$ has the interval of convergence $-1 \le x < 1$. Let $f(x)$ be its sum.

(a) Find $f(0)$ and $f'(0)$.

(b) Justify that the interval of convergence is $-1 \le x < 1$.

1978 BC5
Solution

(a) $f(0)=\dfrac{\ln 1}{1}=0$

$f'(0)=\dfrac{\ln 2}{2}$

(b) $$\lim_{n\to\infty}\left|\frac{\frac{\ln(n+2)}{n+2}x^{n+1}}{\frac{\ln(n+1)}{n+1}x^{n}}\right| = \lim_{n\to\infty}\left|\frac{\ln(n+2)}{\ln(n+1)}\right|\cdot\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|\cdot|x| = 1\cdot 1\cdot|x|$$

The series converges if $|x|<1$. Now check the endpoints.

At $x=-1$,

(i) the series is alternating

(ii) $\dfrac{\ln(n+1)}{n+1}\to 0$ as $n\to\infty$ since $\dfrac{\ln x}{x}\to 0$ as $x\to\infty$

(iii) $\dfrac{\ln(n+1)}{n+1}<\dfrac{\ln n}{n}$ for $n\ge 3$ since $\dfrac{\ln n}{n}$ decreases for $x>e$.

Therefore the series converges for $x=-1$

At $x=1$,

<u>Comparison Test</u>:

$\ln(n+1)>1$ for $n\ge 2$, so $\dfrac{\ln(n+1)}{n+1}>\dfrac{1}{n+1}$ for $n\ge 2$

Therefore the series diverges at $x = 1$ by comparison with the divergent harmonic series.

or

<u>Integral Test</u>:

$$\sum_{n=2}^{\infty}\frac{\ln(n+1)}{n+1}\ge\int_2^{\infty}\frac{\ln(x+1)}{x+1}dx=\lim_{t\to\infty}\int_2^{t}\frac{\ln(x+1)}{x+1}dx=\lim_{t\to\infty}\tfrac{1}{2}(\ln(x+1))^2\Big|_2^t=\infty$$

Therefore the series diverges at $x = 1$.

1978 BC7

A particle moves in the plane so that at any time t, $0 \le t \le 1$, its position is given by $x = \frac{1}{4}e^{8t} - 2t$ and $y = e^{4t}$. Let C denote the path traced by the particle.

(a) Find the components of the velocity vector for any time t.

(b) Find the arc length of C.

(c) Set up an integral, involving only the variable t, that represents the area of the surface generated by rotating C about the y-axis. Do not evaluate the integral.

1978 BC7
Solution

(a) $\dfrac{dx}{dt} = 2e^{8t} - 2$

$\dfrac{dy}{dt} = 4e^{4t}$

(b) Arc length $= \int_0^1 \sqrt{(2(e^{8t} - 1))^2 + (4e^{4t})^2}\, dt$

$= \int_0^1 \sqrt{4e^{16t} - 8e^{8t} + 4 + 16e^{8t}}\, dt = \int_0^1 \sqrt{4e^{16t} + 8e^{8t} + 4}\, dt$

$= \int_0^1 \sqrt{(2e^{8t} + 2)^2}\, dt = \int_0^1 2(e^{8t} + 1)\, dt$

$= 2\left(\frac{1}{8}e^{8t} + t\right)\Bigg|_0^1 = 2\left(\frac{1}{8}e^{8} + 1 - \frac{1}{8} - 0\right) = 2\left(\frac{1}{8}e^{8} + \frac{7}{8}\right)$

$= \frac{1}{4}e^{8} + \frac{7}{4}$

(c) Surface area $= 2\pi \int_0^1 \left(\frac{1}{4}e^{8t} - 2t\right)\sqrt{(2(e^{8t} - 1))^2 + (4e^{4t})^2}\, dt$

$= 2\pi \int_0^1 \left(\frac{1}{4}e^{8t} - 2t\right)(2e^{8t} + 2)\, dt$ (see part (b) for simplification)